MATING NON-RENORMALIZABLE QUADRATIC POLYNOMIALS

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Abstract. In this paper we prove the existence and uniqueness of matings of the basilica with any quadratic polynomial which lies outside of the 1/2-limb of $\mathcal{M}$, is non-renormalizable, and does not have any non-repelling periodic orbits.

1. Introduction

1.1. **Two definitions of mating.** The idea of mating quadratic polynomials was introduced by Douady and Hubbard [**Do2**] as a way to dynamically parameterize parts of the parameter space of quadratic rational maps by pairs of quadratic polynomials. We will present several different ways of describing the construction, which lead to equivalent definitions in the case which is of interest to us.

Consider two quadratic polynomials $f_1(z) = z^2 + c_1$ and $f_2(z) = z^2 + c_2$ whose Julia sets J_1 and J_2 are connected and locally connected. For $i = 1, 2$ denote Φ_i the Böttcher coordinate at infinity

$$\Phi_i : \hat{\mathbb{C}} \setminus K_i \to \hat{\mathbb{C}} \setminus \bar{\mathbb{D}},$$

where K_i is the filled Julia set of f_i. It gives a conjugation

$$\Phi_i \circ f_i(z) = (\Phi_i(z))^2, \text{ for } i = 1, 2.$$

Carathéodory's Theorem implies that Φ_i^{-1} extends to a continuous parameterization $\partial\mathbb{D} \to J_i$. Setting

$$\gamma_i : t \to \Phi_i^{-1}(e^{2\pi i t}) \in J_i,$$

we have

$$f_i(\gamma_i(t)) = \gamma_i(2t). \tag{1.1}$$

The topological space

$$X = (K_1 \sqcup K_2)/(\gamma_1(t) \sim \gamma_2(-t))$$

is obtained by glueing the two filled Julia sets along their boundaries in reverse order. Note that by (1.1) the dynamics of $f_1|_{K_1}$ and $f_2|_{K_2}$ correctly defines a dynamical system $F : X \to X$,

$$F = (f_1\big|_{K_1} \sqcup f_2\big|_{K_2})/(\gamma_1(t) \sim \gamma_2(-t)).$$

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If X is homeomoprhic to S^2, then we say that f_1 and f_2 are *topologically mateable*. In this case, we call the mapping F the *topological mating*, and use the notation $F = f_1 \sqcup_{\mathcal{T}} f_2$.

Assume further, that there exists a homeomorphic change of coordinate $\psi : X \to \hat{\mathbb{C}}$ which is conformal on $\overset{\circ}{K}_1 \cup \overset{\circ}{K}_2$ and such that

$$R = \psi \circ F \circ \psi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

is a rational mapping. We then say that R is a *conformal mating* (or simply a *mating*) of f_1 and f_2, and write $R = f_1 \sqcup f_2$. The pair of quadratics f_1 and f_2 is then called *conformally mateable*. Conformal mateability thus implies, in particular, topological mateability.

Let us give another useful definition of mating. Let © be the complex plane compactified by adjoining the circle of directions at infinity $\{\infty \cdot e^{2\pi i\theta} : \theta \in S^1\}$. Given two quadratic polynomials f_1 and f_2 as before, consider the extension of f_i to the circle at infinity given by

$$f_i(\infty \cdot e^{2\pi i\theta}) = \infty \cdot e^{4\pi i\theta}.$$

Glueing the two circles at infinity in reverse order, we obtain a 2-sphere $\Omega = ©_1 \cup ©_2 / \sim_\infty$, with the equivalence relation $\sim_\infty$ identifying $(\infty \cdot e^{2\pi i\theta_1})$ with $(\infty \cdot e^{2\pi i\theta_2})$ whenever $\theta_1 = -\theta_2$, and a well defined map $f_1 \sqcup_{\mathcal{F}} f_2$ equal to f_i on $©_i$, $i = 1, 2$. The map $f_1 \sqcup_{\mathcal{F}} f_2$ is called the *formal mating* between f_1 and f_2.

For each $\theta \in S^1$ we denote $R_i(\theta)$ the *external ray* of f_i with angle θ given by

$$\Phi_i^{-1}(\{re^{2\pi i\theta} \text{ for } r \geq 1\}).$$

Label $\hat{R}_i(t)$ the closure of $R_i(t)$ in Ω. We define the *ray equivalence relation* $\sim_r$ on Ω in the following way: $x \sim_r y$ if and only if there exists a finite sequence of closed external rays $\{\hat{R}_{i_j}(t_j)\}_{j=1,\dots,k}$ with the property

$$\hat{R}_{i_j}(t_j) \cap \hat{R}_{i_{j+1}}(t_{j+1}) \neq \emptyset, \text{ for } 1 \leq j \leq k-1 \text{ and } \hat{R}_{i_1}(t_1) \ni x,\ \hat{R}_{i_k}(t_k) \ni y.$$

If f_1 and f_2 are topologically mateable then it follows from the definition that the topological space $©_1 \sqcup ©_2 / \sim_\infty$ modulo $\sim_r$ is again a 2-sphere and

$$f_1 \sqcup_{\mathcal{T}} f_2 = f_1 \sqcup_{\mathcal{F}} f_2 / \sim_r .$$

We can now give another equivalent definition of conformal mating in terms of ray equivalence: f_1 and f_2 are *conformally mateable* if there exists a rational mapping $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a pair of semiconjugacies $\phi_i : K_i \to \hat{\mathbb{C}},\ i = 1, 2$

$$R \circ \phi_i = \phi_i \circ f_i,$$

such that the following holds: ϕ_i is conformal on $\overset{\circ}{K}_i$, and $\phi_i(z) = \phi_j(w)$ if and only if $z \sim_r w$. The map R is called a *conformal mating* between f_1 and f_2.

Recall that two branched coverings $F_i : S^2 \to S^2$, $i = 1, 2$ with finite postcritical sets P_i are equivalent in the sense of Thurston if there exist orientation preserving homeomorphisms of the sphere ϕ and ψ such that $\phi \circ F_1 = F_2 \circ \psi$, and ψ is isotopic to ϕ rel P_1. Using Thurston's characterization of postcritically finite rational mappings as branched coverings (see [**DH2**]), Tan Lei [**Tan**] and Rees [**Re1**] demonstrated

that if $f_i(z) = z^2 + c_i$, $i = 1, 2$ is a pair of postcritically finite quadratics and the parameters c_1 and c_2 are not in conjugate limbs of the Mandelbrot set, then the formal mating $f_1 \sqcup_{\mathcal{F}} f_2$ (or a certain degenerate form of it) is equivalent to a quadratic rational map R in the sense of Thurston.

Further, Rees [**Re2**] and Shishikura [**Sh1**] showed that under the above assumptions, f_1 and f_2 are conformally mateable.

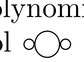
Note that the condition that c_1 and c_2 are not in conjugate limbs is clearly necessary for topological mateability. Indeed, otherwise the cycles of external rays $\{R_1(t_j)\}$ and $\{R_2(s_j)\}$ landing at the dividing fixed points of the respective maps have opposite angles $t_j = -s_j$ (see e.g. [**Mi3**]). Thus $\{\hat{R}_1(t_j)\} \cup \{\hat{R}_2(s_j)\}$ separates Ω and therefore $\Omega/\sim_r$ is not homeomorphic to S^2. It is remarkable that this condition is also sufficient when f_1 and f_2 have finite critical orbits, as this includes cases when both Julia sets are dendrites with empty interior.

First examples of matings not based on Thurston's characterization of rational maps appeared in the paper of Zakeri and the second author [**YZ**]. Before formulating it, recall that an irrational number $\theta \in (0, 1)$ is of *bounded type* if there exists $B > 0$ such that θ can be expressed as an infinite continued fraction with terms bounded by B.

Theorem. *Let θ_1 and θ_2 be two irrationals of bounded type, such that $\theta_1 + \theta_2 \neq 1$. Then the pair of quadratic polynomials $f_i = e^{2\pi i\theta_j} z + z^2$, $j = 1, 2$ are conformally mateable.*

The mating $R = f_1 \sqcup f_2$ is unique up to a Möbius change of coordinates, and is identified algebraically. However, it is very far from being postcritically finite. The postcritical sets of its two critical points are quasicircles, bounding a pair of Siegel disks. The approach taken in [**YZ**] consists in defining a dynamical *puzzle* partition of the Riemann sphere $\hat{\mathbb{C}}$ for the mapping R. The renormalization theory of critical circle maps [**Ya**] can be used to show that nested sequences of puzzle pieces shrink to points. This provides a combinatorial description of the Julia set of R, sufficient to verify that it is a mating.

The history of the problem we consider in this paper is as follows. In 1995 J. Luo [**Luo**] has proposed an approach to constructing a particular class of non postcritically finite matings of the following sort. A quadratic polynomial $f_c(z) = z^2 + c$ is called *starlike* if c is contained in one of the hyperbolic components attached to the main cardioid of the Mandelbrot set $\mathcal{M}$. The name is due to the fact that Hubbard trees associated to such components have only one branching point.

A *Yoccoz' quadratic polynomial* has only repelling periodic cycles, and is renormalizable at most finitely many times. Yoccoz (see e.g. [**Hub**]) has proved that such polynomials are combinatorially rigid, and have locally connected Julia sets. Luo has proposed mating starlike maps with Yoccoz' ones, arguing that the Yoccoz' puzzle partition for quadratics can be transplanted into the quadratic rational map. In this paper we carry this program out for a particular instance of critically finite starlike polynomial $f_{-1}(z) = z^2 - 1$, whose Julia set is known as the *basilica*. We use the symbol  as a graphical reference to this particular quadratic parameter, to

avoid awkward notation. Thus f_{-1} becomes $f_{\circ\bigcirc\circ}$, and its Julia set is denoted $J_{\circ\bigcirc\circ}$. We prove:

Main Theorem. *Suppose c is a non-renormalizable parameter value outside the 1/2-limb of $\mathcal{M}$ such that f_c does not have a non-repelling periodic orbit. Then the quadratic polynomials f_c and $f_{\circ\bigcirc\circ}$ are conformally mateable, and their mating is unique up to a Möbius coordinate change.*

It will be evident from the argument how to adapt it to work for an arbitrary starlike map, however, we decided to specialize to the case $f_{\circ\bigcirc\circ}$ for the sake of clarity. Potentially, the methods of the proof should also work for the case of a general Yoccoz' parameter c, or even an infinitely renormalizable parameter with good combinatorics.

Since $f_{\circ\bigcirc\circ}$ has a superattracting orbit $0 \to -1 \to 0$, any candidate mating R must exhibit a superattracting orbit of order 2. Let us place the critical point at ∞ and assume that $R(\infty) = 0$, $R^2(\infty) = \infty$. The following family will serve as our candidate matings:

$$R_a(z) = \frac{a}{z^2 + 2z}.$$

The critical points of R_a are ∞ and -1.

A crucial obstacle now (and a principal difference with [**YZ**]) is that there is no algebraic approach to specifying the candidate mating of f_c and $f_{\circ\bigcirc\circ}$. Instead, and similarly to Yoccoz' rigidity result, we will define a puzzle partition in the parameter space of R_a, and select the mating as the unique intersection point of a specific sequence of puzzle-pieces.

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2. Basic properties for R_a and $f_{\circ\bigcirc\circ}$

For ease of reference, we summarize in this section some of the basic properties of the mapping $f_{\circ\bigcirc\circ}(z) = z^2 - 1$ and the quadratic rational maps in the family R_a. We refer the reader to [**Mi1**] for the discussion of the properties of Fatou and Julia sets, and to [**Mi3**] for the properties of external rays of polynomial maps.

2.1. Basic properties of $f_{\circ\bigcirc\circ}$.

Let us begin with the following general statement (cf. [**Mi1**]).

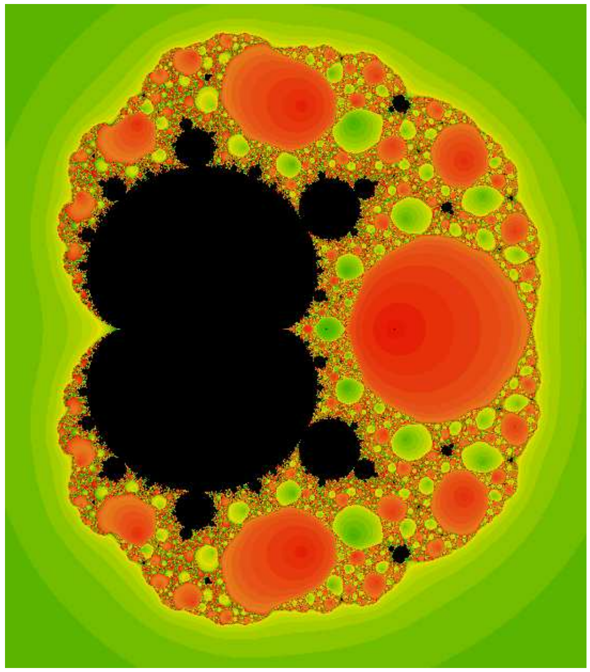

FIGURE 1. The parameter set for R_a.

Lemma 2.1. *Let U be a simply-connected immediate basin of a superattracting periodic point of a rational mapping $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of period q. Denote $\phi : U \mapsto \mathbb{D}$ a Böttcher coordinate: $\phi(F^q(z)) = (\phi(z))^d$ for some $d > 1$. An internal ray is a curve $\phi^{-1}(\{re^{2\pi it} | \, r \in [0, 1)\})$. Then:*

- *suppose, p is a repelling or parabolic periodic point on the boundary of U. Then p is the landing point of an internal ray whose period is divisible by the period of p;*
- *conversely, every periodic internal ray lands at a repelling or parabolic periodic point in ∂U.*

Let B_0, B_{-1} be the immediate basins of attraction of 0 and -1 respectively for $f_{\circ\!\bigcirc\!\circ}$. Let B_∞ be the basin of attration at infinity. Note that $f_{\circ\!\bigcirc\!\circ} : B_0 \mapsto B_{-1}$ is also a $2 \to 1$ covering branched at 0.

Lemma 2.2. *For any two Fatou components A and B of $f_{\circ\!\bigcirc\!\circ}$, neither of which is the attracting basin of infinity, exactly one of the following holds:*

(1) $\overline{A} \cap \overline{B} = \emptyset$.
(2) $\overline{A} \cap \overline{B}$ *is only one point, which is a pre-fixed point for* $f_{\circ\!\bigcirc\!\circ}$.
(3) $A = B$.

The statement of the Lemma follows immediately from the Maximum Principle. Note, that the boundaries of the Fatou components B_0 and B_{-1} touch at the repelling fixed point α of $f_{\circ\!\bigcirc\!\circ}$.

Since the mapping $f_{\circ\bigcirc\circ}$ is hyperbolic, its Julia set is locally connected. In particular, if $\Phi : \hat{\mathbb{C}} \setminus K(f_{\circ\bigcirc\circ}) \mapsto \mathbb{C} \setminus \mathbb{D}$ denotes the Böttcher coordinate at ∞, the Carathéodory's Theorem implies that Φ^{-1} extends continuously to $\partial\mathbb{D}$. Moreover, every external ray $R(\theta) = \Phi^{-1}(\{re^{2\pi i\theta} | \, r > 1\})$ lands at a point of the Julia set. We denote

$$\gamma(\theta) = \lim_{r \to 1^+} \Phi(re^{2\pi i\theta}).$$

Hyperbolicity of $f_{\circ\bigcirc\circ}$ also implies:

Lemma 2.3. *Let F_i be an arbitrary infinite sequence of distinct Fatou components of $f_{\circ\bigcirc\circ}$. Then* $\operatorname{diam} F_n \to 0$.

We will also make use of the following Lemma:

Lemma 2.4. *A point $z \in J_{\circ\bigcirc\circ}$ is a landing point of precisely two external rays if and only if z is a preimage of the fixed point α. No other point $z \in J_{\circ\bigcirc\circ}$ is biaccessible.*

The angles of the two external rays which land at α are easily identified as $1/3$ and $2/3$.

2.2. **Properties of maps in the family R_a.** In what follows, we will refer to the illustration of the parameter space for the family R_a pictured in Figure 1.

For R_a let A_∞ be the immediate basin of attraction at infinity, and A_0 the Fatou component containing 0.

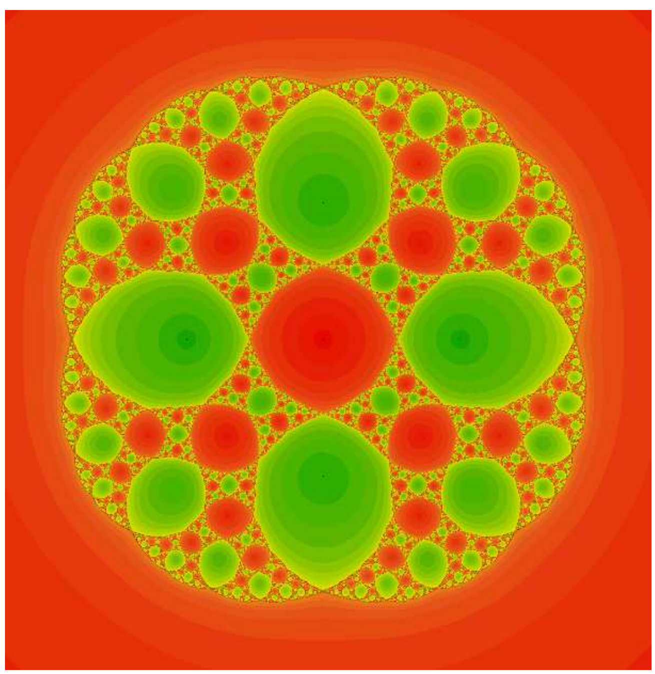

FIGURE 2. A capture dynamics: dynamical plane of R_2.

Let us note:

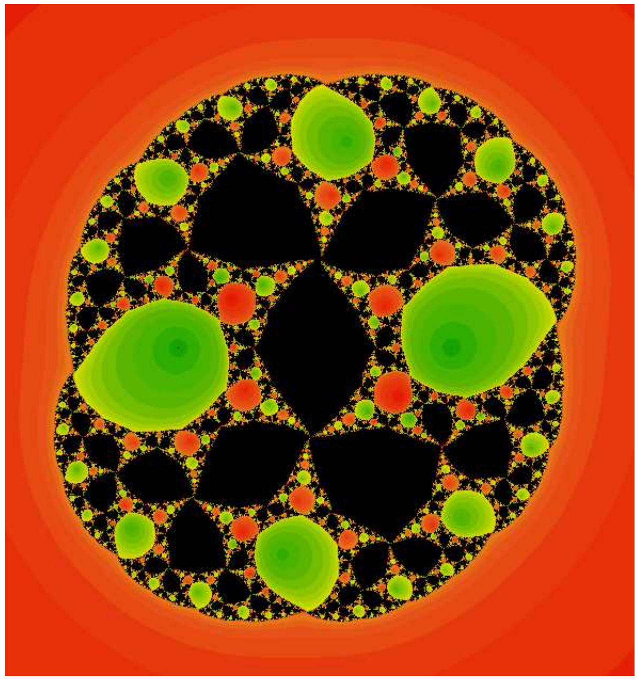

FIGURE 3. Stony Brook preprint cover: the dynamical plane of the mating of basilica and Douady's rabbit.

Proposition 2.5. *The Fatou components A_0 and A_∞ are distinct and simply-connected. The critical point -1 of R_a is never contained in A_∞.*

Proof. We have $A_0 \neq A_\infty$ by Denjoy-Wolff Theorem. If A_∞ is multiply-connected, then, necessarily, $-1 \in A_\infty$, by the Riemann-Hurwitz formula. Thus A_0 contains all critical values of R_a. In this case, it follows (see e.g. [**Mi2**], Lemma 8.1) that the Julia set of R_a is totally disconnected, and that every orbit in the Fatou set converges to an attracting fixed point, which is impossible. □

Note, that whenever a is such that $-1 \in A_0$, the Fatou set of R_a is the union of A_0 and A_∞. The Julia set $J(R_a)$ is the common boundary of the two Fatou components, and we have (see, for instance, [**CG**], Theorem 2.1 on p. 102):

Proposition 2.6. *If $-1 \in A_0$, then $J(R_a)$ is a quasicircle.*

In the parameter space (Figure 1) the above values of a form the "exterior" hyperbolic component which we denote P_∞.

More generally, a *capture* hyperbolic component for the family R_a contains maps for which there exists an iterate $R_a^n(-1) \in A_\infty$. The smallest such n will be referred to as the *generation* of the capture component.

For instance, $a = 2$ is the center of the biggest red "bubble" in Figure 1, in which we have $R_a^2(-1) \in A_\infty$. The corresponding Julia set is depicted in Figure 2. Similarly to the statement of Lemma 2.2, we will show in §5:

Lemma 2.7. *Suppose that the parameter a is chosen outside of the closure $\bar{P}_\infty$. Then given any two Fatou components A and B in the basin of ∞ of R_a exactly one of the following holds:*

(1) $\overline{A} \cap \overline{B} = \emptyset$,
(2) $\overline{A} \cap \overline{B}$ *is only one point,*
(3) $A = B$.

Moreover, if the case (2) occurs, then $\bar{A} \cap \bar{B}$ is either a preimage of the fixed point

$$x_a \equiv \bar{A}_0 \cap \bar{A}_\infty$$

or a pre-critical point. For the latter possibility to occur, the parameter a must belong to the boundary of a capture component.

Denote $\mathcal{M}at$ the set of parameter values a not contained in any of the capture components. This set is colored in black in Figure 1. The interior of $\mathcal{M}at$ contains matings with basilica, and thus should be naturally identified with $\mathring{\mathcal{M}}$ with the 1/2-limb removed.

As an example of a mating in $\mathcal{M}at$, consider Figure 3. This image was popularized on the cover of Stony Brook preprint series; it is the mating of Douady's rabbit with basilica.

3. Orbit portraits for quadratic polynomials

In this section we provide a brief summary of several results on the combinatorics of external rays of quadratic polynomials following Milnor's paper [**Mi3**]. All proofs are given in [**Mi3**].

Let the points $\{x_1, x_2 = f(x_1), \ldots, x_p = f(x_{p-1})\}$ form a periodic orbit of a quadratic polynomial $f_c(z) = z^2 + c$ with period p. Assume further, that this orbit is either repelling or parabolic, and hence the landing set of a finite collection of periodic external rays $R(\theta_i)$ (see e.g. [**Mi1**]).

Definition 3.1. For each $1 \leq i \leq p$ let $A_i = \{\theta_1^i, \ldots, \theta_k^i\}$ denote the set of angles of the external rays landing at x_i. The collection $\mathcal{O} = \{A_1, \ldots, A_p\}$ is called the *orbit* portrait of the cycle $(x_1, \ldots, x_p)$. According to the type of the cycle, the orbit portrait is either *repelling* or *parabolic*.

Given the periodicity of x_i, the iterate f_c^i permutes the rays with angles in A_i. The following is immediate:

Lemma 3.1. *Given an orbit portrait $\mathcal{O} = \{A_1, \ldots, A_p\}$ the size of A_i is the same for all i. Moreover, $A_{i+1} = 2A_i \bmod \mathbb{Z}$, and if $|A_i| \geq 3$, then the cyclic order of the angles $\theta_j^i \in A_i$ is the same as that of their images $2\theta_j^i \bmod \mathbb{Z} \in A_{i+1}$.*

Definition 3.2. For $A = \{\theta_1, \ldots, \theta_k\} \subset \mathbb{T}$, write $\exp(A) = \{e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_k}\} \subset S^1$. A *formal orbit portrait* is a collection $\{A_1, \ldots, A_p\}$ of subsets of $\mathbb{T}$ for which the following properties hold:

- each A_i is a finite subset of $\mathbb{T}$;

- for each j modulo p, the doubling map $t \mapsto 2t \bmod \mathbb{Z}$ carries A_j bijectively onto A_{j+1} preserving the cyclic order around the circle;
- all of the angles in $A_1 \cup \cdots \cup A_p$ are periodic under doubling with the same period rp;
- for each $i \neq j$, the convex hulls of the sets $\exp(A_i)$ and $\exp(A_j)$ are disjoint.

The *valence* of an orbit protrait $\mathcal{O}$ is $v_{\mathcal{O}} = |A_i|$. Every angle in A_i is periodic of period pr. Since there are $pv_{\mathcal{O}}$ angles in $\mathcal{O}$, the quantity $v_{\mathcal{O}}/r$ is the number of distinct cycles of external rays in the orbit portrait $\mathcal{O}$.

Lemma 3.2. *Only two possibilities can occur: either $v_{\mathcal{O}} = r$ or $v_{\mathcal{O}} = 2$ and $r = 1$.*

Assume that $v_{\mathcal{O}} \geq 2$. For each A_i, the complement $\mathbb{T} \setminus A_i$ consists of finitely many *complementary arcs*. Each such arc corresponds to a sector between two of the rays landing at x_i.

Lemma 3.3. *Let $\mathcal{O} = \{A_1, \ldots, A_p\}$ be a formal orbit portrait. Then every complementary arc for A_i, except for one is mapped one-to-one under $z \mapsto 2z$ onto a complementary arc of A_{i+1}. The exception is the* critical arc *of A_i, which has length greater than $1/2$. The image of the critical arc wraps around the whole unit circle, covering one of the complementary arcs of A_{i+1} twice.*

If the portrait $\mathcal{O}$ is realized by a quadratic polynomial, then for each i, the sector corresponding to the critical arc of A_i contains the critical point 0.

Lemma 3.4. *Assume that $v_{\mathcal{O}} \geq 2$. There exists a unique shortest complementary arc in $\mathcal{O}$. If the portrait is realized by a quadratic polynomial f_c, then the sector corresponding to this arc can be characterized among the pv sectors formed by the rays landing at points x_i as the one which contains the critical value $c = f_c(0)$ and no points of the orbit x_i.*

Definition 3.3. The complementary arc in the previous lemma is referred to as the *characteristic arc* of the orbit portrait.

4. Bubble rays

To construct a Yoccoz puzzle partition for the quadratic rational maps in $\mathcal{M}at$, we will use chains of Fatou components in place of external rays. This method was employed in [**YZ**] and [**Ro2**], it was also suggested in [**Luo**]. We begin by describing such chains in the filled Julia set of $f_{\circ\bigcirc\circ}$; this discussion, while mostly trivial, will serve as a useful preparation for handling maps in the family R_a.

4.1. **Bubble rays for $f_{\circ\bigcirc\circ}$.** Recall that B_0 and B_{-1} denote the components of the immediate super-attracting basin of $f_{\circ\bigcirc\circ}$, labelled according to the point in the critical orbit they surround.

Definition 4.1. A *bubble* of $K_{\circ\bigcirc\circ}$ is a Fatou component $F \subset \mathring{K}_{\circ\bigcirc\circ}$. The *generation* of a bubble F is the smallest non-negative $n = \mathrm{Gen}(F)$ for which $f^n_{\circ\bigcirc\circ}(F) = B_0$. The *center* of a bubble F is the preimage $f_{\circ\bigcirc\circ}^{-\mathrm{Gen}(F)}(0) \cap F$.

If $F \neq B_0$, then let G be the bubble with the lowest value of $\mathrm{Gen}(G)$ for which $\bar{G} \cap \bar{F} \neq \emptyset$. We will refer to G as the *predecessor* of F, and to the point $x = \mathrm{root}(F) \equiv \bar{G} \cap \bar{F}$ as the *root* of F.

A bubble ray $\mathcal{B}$ is a collection of bubbles $\cup_0^{m \leq \infty} F_k$ such that for each k the intersection $\overline{F_k} \cap \overline{F_{k+1}} = \{x_k\}$ is a single point, and $\mathrm{Gen}(F_k) < \mathrm{Gen}(F_{k+1})$.

Note that by Lemma 2.2, each of the points x_k is a preimage of the α-fixed point of $f_{\circ\bigcirc\circ}$. If $m < \infty$, we will refer to the component F_m as the *last bubble* of $\mathcal{B}$. Hyperbolicity of $f_{\circ\bigcirc\circ}$ readily implies:

Proposition 4.1. *There exist $s \in (0,1)$, and $C > 0$ such that for a bubble $F \subset \mathring{K}_{\circ\bigcirc\circ}$ we have*

$$\mathrm{diam}(F) \leq C s^{\mathrm{Gen}(F)}.$$

In particular, for each infinite bubble ray $\mathcal{B} = \cup_0^\infty F_k$ there exists a unique point $x \in J_{\circ\bigcirc\circ}$ such that $F_k \to x$ in Hausdorff sense.

We refer to x as the *landing point* of $\mathcal{B}$. By Lemma 2.2 we have:

Proposition 4.2. *If two bubble rays $\mathcal{B}_1$, $\mathcal{B}_2$ have the same landing point, then one of them is contained in the other one.*

By Lemma 2.1, each pre-periodic point on the boundary of a bubble is a landing point of an internal ray. We may therefore define:

Definition 4.2. The *axis* of a bubble ray $\mathcal{B} = \{F_k\}_0^{m \leq \infty}$ is the closed union

$$\gamma(\mathcal{B}) \equiv \overline{\cup_0^m \gamma_k},$$

where γ_k for $k \geq 1$ is the union of two internal rays of F_k connecting its center to the points x_{k-1} and x_k, and γ_0 is the internal ray of F_0 terminating at x_0.

Let x be the landing point of an infinite bubble ray $\mathcal{B}$. As the Julia set $J_{\circ\bigcirc\circ}$ is locally connected, Carathéodory's Theorem implies that there exists at least one external ray $R(\theta)$ landing there. By Lemma 2.4, such θ is unique. Let us refer to the number $-\theta$ as the *angle* of the bubble ray $\mathcal{B}$ and denote it

$$\measuredangle(\mathcal{B}) \equiv -\theta.$$

By Proposition 4.2, $\measuredangle(\mathcal{B}_1) = \measuredangle(\mathcal{B}_2)$ implies that one of these rays is a subset of the other.

We will call a bubble ray $\mathcal{B}$ *periodic* if the angle $\measuredangle(\mathcal{B})$ is periodic under doubling; the *period* of the ray will refer to the period of its angle.

Note that the angle of a bubble ray can be determined intrinsically, from the choice of the bubbles themselves. Indeed, consider the *spine*

$$\ell(K_{\circ\bigcirc\circ}) \equiv K_{\circ\bigcirc\circ} \cap \mathbb{R} = [-\beta, \beta],$$

where β is the non-dividing fixed point of $f_{\circ\bigcirc\circ}$. The spine may also be seen as the union of the axes of the bubble rays $\mathcal{B}_+$, $\mathcal{B}_-$ starting with the bubble B_0 and terminating at $\pm\beta$ respectively.

Let $\mathcal{B} = \cup F_k$ be an infinite bubble ray, landing at $x \neq \beta$. Consider the forward iterates $x_k = f_{\infty\!\bigcirc\!\infty}^k(x)$. Define a sequence $s(\mathcal{B}) = (s_i)_1^\infty$ of 0's and 1's as follows. We set

- $s_i = 0$ if x_i is above the spine, or equivalently, if there is a bubble F_k with $k \geq i$ which is above the spine;
- $s_i = 1$ if x_i is below the spine, or equivalently, if there is a bubble F_k with $k \geq i$ which is below the spine;
- if i is the first instance when neither of these two possibilities holds, set $s_i = 1$, and $s_j = 0$ for all $j > i$ (note, that in this case we necessarily have $x_i = -\beta$.

For $\mathcal{B} \subset \mathcal{B}_+$ we set $s(\mathcal{B}) = (0)_0^\infty$.
We will sometimes refer to the dyadic sequence $s(\mathcal{B})$ as the *intrinsic address* of $\mathcal{B}$. Noting that

$$(\beta, +\infty) = R(0), \text{ and } (-\infty, -\beta) = R(1/2),$$

we immediately have

Proposition 4.3. *For each infinite bubble ray $\mathcal{B}$ we have*

$$\measuredangle(\mathcal{B}) = -\sum_{i=1}^{\infty} 2^{-i} s_i, \text{ where } s(\mathcal{B}) = (s_i)_0^\infty.$$

4.2. **Bubble rays for a map R_a.** The definition of a bubble ray for a rational mapping R_a is completely analogous to Definition 4.1.

Definition 4.3. A *bubble* of R_a is a Fatou component $F \subset \cup R_a^{-k}(A_\infty)$. The *generation* of a bubble F is the smallest non-negative $n = \operatorname{Gen}(F)$ for which $R_a^n(F) = A_\infty$. The *center* of a bubble F is the preimage $R_a^{-\operatorname{Gen}(F)}(\infty) \cap F$.

A bubble ray $\mathcal{B}$ is a collection of bubbles $\cup_0^{m\leq\infty} F_k$ such that for each k the intersection $\overline{F_k} \cap \overline{F_{k+1}} = \{x_k\}$ is a single point, and $\operatorname{Gen}(F_k) < \operatorname{Gen}(F_{k+1})$.

The structure of bubble rays for R_a is particularly easy to describe when $a \in \mathcal{M}at$, and somewhat more difficult in the capture case. We consider the simpler possibility first.

The case $a \in \mathcal{M}at$ Consider the Böttcher coordinates $b_1 : \mathcal{D} \to B_0$, and $b_2 : \mathcal{D} \to A_\infty$. The identification

$$\phi \equiv b_2 \circ b_1^{-1} : B_0 \to A_\infty$$

conjugates the dynamics of $f_{\infty\!\bigcirc\!\infty}$ and R_a. Note that by Lemmas 2.1 and 2.7 the components A_∞ and A_0 have a single common boundary point $x = \lim_{r\to 1-} b_2(r)$ and is fixed by the dynamics of R_a. By Lemma 2.7 we have the following:

Proposition 4.4. *If two bubbles F_1 and F_2 of R_a touch at a boundary point z, then z is a preimage of x.*

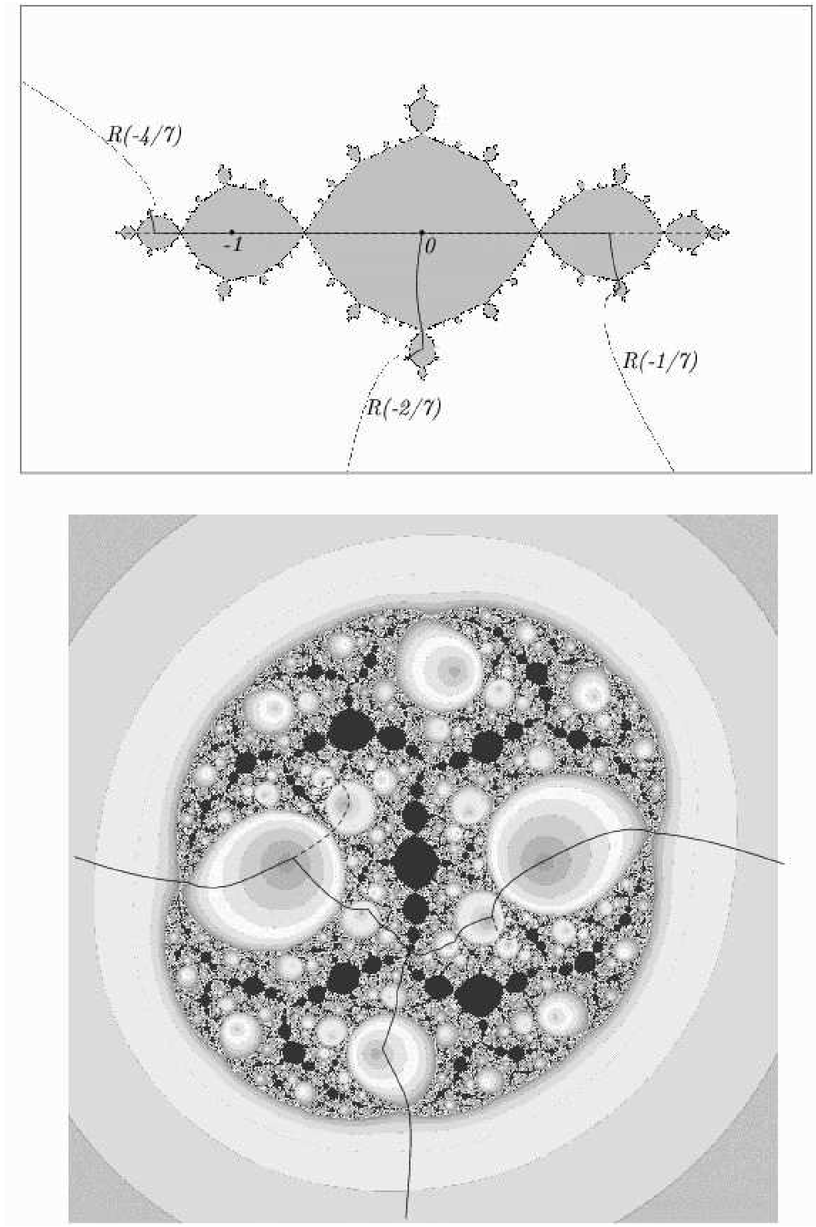

FIGURE 4. Bubble rays for $f_{\circ\bigcirc\circ}$ and R_a. The picture below is a mating of $f_{\circ\bigcirc\circ}$ with a hyperbolic parameter in the 1/3-limb of $\mathcal{M}$. Three periodic bubble rays land at a repelling fixed point of the rational map. The solid lines follow their axes. Their angles are 1/7, 2/7, and 4/7 respectively. The axes of the same bubbles are shown inside $K_{\circ\bigcirc\circ}$ in the above pictures. The broken lines show the position of the spines.

By Lemma 2.1, the axis $\gamma(\mathcal{B})$ of a bubble ray $\mathcal{B}$ of R_a can be defined as before. To define the *spine* ℓ_a begin by considering the union of internal rays $l_\infty \subset \hat{\mathbb{C}}$ which is the image under b_2 of the segment $(-1, 1)$. Let $l_0 \subset A_0$ be its preimage, and set

$$t_1 = \bar{l}_\infty \cup \bar{l}_0 \cup \{x\}.$$

We now inductively define $t_n = t_{n-1} \cup h_1 \cup h_2$ where h_i are the two components of $R_a^{-1}(t_{n-1}) \setminus A_\infty$ intersecting t_{n-1}.

Definition 4.4. We set

$$\ell_a = \cup t_n,$$

and endow this arc with positive orientation as induced by the orientation of $(-1, 1) \mapsto l_\infty$. Further, for a bubble F of R_a with $F \cap \ell_a = \emptyset$, we say that F is *above* the spine, if the unique finite bubble ray connecting it to the spine lies above ℓ_a with respect to the orientation of ℓ_a. In the complementary case, we say that the bubble F is *below* the spine.

We define the *intrinsic address* $s(\mathcal{B})$ of a bubble ray $\mathcal{B}$ in exactly the same fashion as before.

The oriented spine allows us to extend inductively the conjugacy $\phi : f_{\circ\!\bigcirc\!\circ}^{-n}(B_0) \to R_a^{-n}(A_\infty)$ so that:

Proposition 4.5. *Denote*

$$L = \mathring{K}_{\circ\!\bigcirc\!\circ} \cup \left(\bigcup_{n=0}^{\infty} f_{\circ\!\bigcirc\!\circ}^{-n}(\alpha) \right).$$

Then ϕ extends as a conjugacy to the whole of L. Moreover, this conjugacy obeys the property:

$$s(\phi(\mathcal{B})) = s(\mathcal{B})$$

for each bubble ray $\mathcal{B}$ in $K_{\circ\!\bigcirc\!\circ}$.

Definition 4.5. For an infinite bubble ray $\mathcal{B}$ of R_a we set the *angle of* $\mathcal{B}$ equal to

$$\measuredangle(B) \equiv \measuredangle(\phi^{-1}(\mathcal{B})).$$

By construction, we have

$$\measuredangle(\mathcal{B}) = \sum_{n=1}^{\infty} 2^{-n} s_n, \text{ where } s(\mathcal{B}) = (s_n)_1^\infty \tag{4.1}$$

for each bubble ray $\mathcal{B}$ of R_a.

The case when a belongs to a capture component. Let us exclude the trivial possibility when the critical value $-a = R_a(-1) \in A_\infty$, and denote $n > 1$ the smallest natural number for which $R_a^n(-1) \in A_\infty$ holds. The conjugacy ϕ can still be extended consistently with the orientation to $f_{\circ\!\bigcirc\!\circ}^{-(n-1)}(B_0)$. Denote $F \ni -a$ the bubble of R_a containing the critical value, and set $H = \phi^{-1}(F) \subset \mathring{K}_{\circ\!\bigcirc\!\circ}$.

Definition 4.6. We define an equivalence relation $\sim$ on $\mathring{K}_{\circ\bigcirc\circ}$ as follows. Connect the two preimages H_1, H_2 of H by a simple arc $h \subset \hat{\mathbb{C}} \setminus K_{\circ\bigcirc\circ}$. The equivalence relation identifies any two bubbles $G_1, G_2 \subset \mathring{K}_{\circ\bigcirc\circ}$ if there exists $l \geq 0$ such that G_1 is connected to G_2 by a component of $f_{\circ\bigcirc\circ}^{-l}(h)$. For points $x_i \in G_i$ we set $x_1 \sim x_2$ if this happens, and if $f_{\circ\bigcirc\circ}^{n+l}(x_1) = f_{\circ\bigcirc\circ}^{n+l}(x_2)$.

One readily verifies:

Lemma 4.6. *In the capture case, the mapping ϕ extends as a surjective conjugacy from*

$$\left(\mathring{K}_{\circ\bigcirc\circ} \cup \bigcup_{i=0}^{\infty} f_{\circ\bigcirc\circ}^{-i}\alpha\right) \Big/_{z_1 \sim z_2} \longrightarrow \bigcup_{i=0}^{\infty} R_a^{-i}(A_\infty \cup \{x\}).$$

5. Parabubble rays.

Removing the α-fixed point from the basilica $K_{\circ\bigcirc\circ}$ separates it into two connected components. We will denote them $\mathcal{L}$ for "left", and $\mathcal{R}$ for "right". Put $\mathcal{R}_e = \mathcal{R} \setminus \overline{B}_0$, (the subscript e, standing for "exterior" of the right half of the basilica). As we will see below, there is a natural correspondence between the components of the interior of $\mathcal{R}_e$, and the capture hyperbolic components in the parameter plane of the family R_a.

For the remainder of this section, let us fix the notation $R_a(z) = R(z,a)$, $R_a^n(z) = R^n(z,a)$.

Definition 5.1. Let a_0 be such that $R_{a_0}^k(-1) = \infty$ for some $k \in \mathbb{N}$, and let n be the smallest such value of k. Then a connected set P of parameters a containing a_0, such that $R^n(-1,a) \in A_\infty$ is called a *capture* hyperbolic component or a *parabubble*. The point a_0 is called a *center* of P. We will see further that it is unique.

Finally, we say that the *generation* of P is n, and write $\mathrm{Gen}(P) = n$.

Set $\xi_n(a) = R^n(-1,a)$. Then we have

$$\xi_{n+1}(a) = \frac{a}{(\xi_n(a))^2 + 2\xi_n(a)} = \frac{a}{\xi_n(a)(\xi_n(a)+2)}. \tag{5.1}$$

From (5.1) it follows by a straightforward induction, that

Lemma 5.1. *The degree of ξ_n is the nearest integer value to $2^{n+1}/3$.*

We now state:

Lemma 5.2. *For $n \geq 2$, the degree of ξ_n is equal to the number of bubbles of generation n in the basilica which are contained in $\mathcal{R}$.*

Proof. To each bubble $B \subset K_{\circ\bigcirc\circ}$ we associate an interval $(a,b) = I_B \subset \mathbb{R}/\mathbb{Z}$, where a, b are the angles of the external rays meeting at the root of B. It is easy to see that the centers of the intervals I_B of all bubbles of generation n are symmetrically distributed around the unit circle and that each I_B does not intersect $1/3$ or $-1/3$. It is easy to verify that the closest integer to $2^n \times (2/3)$ is equal to the number of

I_B which are contained in the interval $(-1/3, 1/3)$. The claim follows from Lemma 5.1. $\square$

Denote A_∞^a the set A_∞ for the map R_a. Let

$$\Phi_a : A_\infty^a \mapsto \hat{\mathbb{C}} \setminus \mathbb{D}$$

be the Böttcher coordinate for R_a normalized so that $\Phi'_a(\infty) > 0$. Note that Φ_a is analytic in a. A direct calculation implies

$$\Phi_a(z) = \frac{1}{2}z + o(1), \text{as } z \to \infty. \tag{5.2}$$

If $-a \in A_\infty$ then Φ_a can be extended around ∞ until we hit a critical point $z = 1 \pm \sqrt{1-a}$ for R_a^2. However, the Green's function $g(z, a) = \log |\Phi_a(z)|$ is still well defined on A_∞ and moves continuously with a, and $g(z, a) \to 0$ as $z \to \partial A_\infty$ for all $a \in \mathbb{C}$. Let P_∞ be the open set of parameters where $-a \in A_\infty$, that is, where $J(R_a)$ is a quasicircle. This capture component obviously contains an open neighborhood of ∞.

By the λ-Lemma of [**MSS**] we have:

Lemma 5.3. *The Julia set $J(R_a)$ moves holomorphically for all $a \in P_\infty$.*

Let us continuously extend the Green's function $g(z, a)$ on the whole sphere so $g(z, a) = 0$ outside A_∞. The proof of Theorems III.3.2 in [**CG**] can be easlily adjusted to the family $R_a^2 : A_\infty \mapsto A_\infty$, to show that the Green's function g is uniformly Hölder α-continuous for $|a| \le C$, some $\alpha = \alpha(C) \in (0, 1]$. As a consequence, $g(-a, a) \to 0$ as $a \to \partial P_\infty$, (see Theorem III.3.3 [**CG**]). Moreover, by (5.2), the function $\Phi_a(-a)$ has a simple pole at ∞. Since $g(-a, a) \to 0$ as $a \to \partial P_\infty$, the Argument Principle implies that $\Phi_a(-a)$ takes every value in $\hat{\mathbb{C}} \setminus \mathbb{D}$ exactly once. We get the following:

Lemma 5.4. *The set $P_\infty \cup \{\infty\}$ is simply connected and P_∞^c has logarithmic capacity equal to $1/2$.*

It is easy to verify that A_∞ does not necessarily move continuously at ∂P_∞ if we step inside P_∞ (e.g. at $a = 3$), but the following holds.

Lemma 5.5. *The set $\overline{A}_\infty$ moves holomorphically for all parameters $a \in (\overline{P}_\infty)^c$. We have $a \in \partial P_\infty$ if $-a \in \partial A_\infty^a$.*

Proof. Put $\psi_a = \Phi_a \circ \Phi_{a_0}^{-1}$. Then ψ_a maps $A_\infty^{a_0}$ onto A_∞^a. If $a \notin \overline{P}_\infty$ then $-a \notin A_\infty$ by definition and we have that $\psi_a(z) = \psi(z, a)$ is a holomorphic motion on $A_\infty^{a_0} \times \overline{P}_\infty^c$. By the Λ-Lemma, ∂A_∞^a also moves holomorphically.

If $-a_1 \in \partial A_\infty$ for some $a_1 \notin \overline{P}_\infty$ then since A_∞ moves holomorphically, the point $-a_1$ is an image of some point $z_1 \in \partial A_\infty^{a_0}$ under ψ, i.e. $\psi(z_1, a_1) = -a_1$. The analytic function $\psi_{z_1}(a)$ satisfies $\psi_{z_1}(a_1) + a_1 = 0$. Either $\psi_{z_1}(a) + a \equiv 0$ or not. If so, then $-a \in \partial A_\infty$ for all $a \in (\overline{P}_\infty)^c$, which is clearly false. If not so, then choose a small disk $B(a_1, \varepsilon) \subset (\overline{P}_\infty)^c$ and some $z_2 \in A_\infty^{a_0}$, with z_2 sufficiently close to z_1, such that $|\psi_{z_1}(a) - \psi_{z_2}(a)| < |\psi_{z_1}(a) + a|$ for $a \in \partial B(a_1, \varepsilon)$. By Roche's Theorem,

$\psi_{z_2}(a)+a=0$ must have a solution $b\in B(a_1,\varepsilon)$, which means that $-b\in A^b_\infty$, which is a contradiction.

$\square$

Corollary 5.6. *The statement of Lemma 2.7 holds for $a\in(\bar{P}_\infty)^c$. Moreover, for every such a, the bubbles of R_a have locally connected boundaries.*

Proof. Consider a mapping R_a with the parameter $a\in(\bar{P}_\infty)^c$ contained in a capture component. Since R_a is a hyperbolic mapping, the boundary of every A^a_∞ is locally connected by the standard considerations. The second claim follows. The first claim is now immediate. $\square$

5.1. **Internal parameter rays.** If P is a capture component of generation $n\geq 1$, for $t\in P$ let $g_n(t)=\Phi_t(R^n(-1,t)))$, so that g_n maps $a\in P$ to the Böttcher coordinate for $R^n_a(-1)$ in A_∞. The function ξ_n a rational function and has a pole of finite order at the center of every capture component (later we show that it is in fact a simple pole). We proceed with the following definition.

Definition 5.2. An *internal parameter ray of angle θ* is a connected component of the set

$$\{g_n^{-1}(re^{2\pi i\theta}):r>1\}.$$

Lemma 5.7. *Let P be a parabubble with* $\mathrm{Gen}(P)=n\geq 2$*, and let $\theta\in\mathbb{T}$ be periodic (pre-periodic) under doubling. Then an internal parameter ray of P with angle θ lands at a point $a_0\in\partial P$. Moreover, the point*

$$p(a_0)=R^n_{a_0}(-1)$$

is a repelling periodic (pre-periodic) point on the boundary of A_∞.

Proof. To fix the ideas, we assume that $\theta=0$ so that $p(a_0)$ is the repelling fixed point where A_∞ and A_0 touch. Set

$$\gamma_n(t)=g_n^{-1}(te^{2\pi i\theta}),\ \text{for } t>1,$$

where we assume that $g_n^{-1}(te^{2\pi i\theta})$ belongs to a chosen connected component of $\{g_n^{-1}(re^{2\pi i\theta}):r>1\}$. We want to show that $\lim\limits_{r\to 1^+}\gamma_n(r)$ exists and is equal to a_0. First note that

$$|\Phi_a^{-1}(r)-p(a)|\leq\delta(r), \tag{5.3}$$

where $\delta(r)\to 0$ as $r\to 1$, which follows by Lemma 2.1. Also, note that the left hand side of (5.3) is a continuous function of both a and r on $\overline{P}\times(1,\infty)$. This implies that $\Phi_a^{-1}(r)\to p(a)$ uniformly as $r\to 1$ on $\overline{P}$.

Therefore, for $a=\gamma_n(r)$,

$$|\Phi^{-1}_{\gamma_n(r)}(r)-p(\gamma_n(r))|\leq\delta(r), \tag{5.4}$$

where $\delta(r)\to 0$ as $r\to 1$. Now, for $|a-a_0|\leq\varepsilon$, we have $|R^n_a(-1)-p(a)|\leq\varepsilon'(\varepsilon)\to 0$, as $\varepsilon\to 0$. On the other hand, since the zeros of $|R^n_a(-1)-p(a)|$ are isolated, we can find a $C>0$ such that if $0<\varepsilon\leq|a-a_0|\leq C$, then $|R^n_a(-1)-p(a)|\geq\varepsilon'$.

If $\gamma_n(r)$ does not land at a_0, take an $a \in \gamma_n(r) \setminus B(a_0, \varepsilon)$, where r is sufficiently close to 1, so that (5.4) holds for $\delta(r) \leq \varepsilon'/2$. But since $|a - a_0| \geq \varepsilon$ we have $|R_a^n(-1) - p(a)| \geq \varepsilon'$, for $a = \gamma_n(r)$, which is a contradiction. Hence $\gamma_n(t)$ must land at a_0. $\square$

The landing property for periodic parameter rays in P_∞ follows from the standard theory in e.g. [**CG**], Theorem 5.2:

Proposition 5.8. *If θ is rational then the internal parameter ray of angle θ in P_∞ lands at a parameter $a \in \partial P_\infty$. Moreover, if $\theta \neq 0$ is periodic then R_a has a parabolic cycle and if θ is strictly preperiodic then R_a is a postcritically finite map.*

Consider the conjugacy ϕ from Lemma 4.6. We have the following:

Lemma 5.9. *Let P be a parabubble of generation $n \geq 2$ and address σ.*

- (I) *There exists a unique bubble $W \in K_{\circ\bigcirc\circ}$ such that the following holds. Let $a \in P$ and denote B_a the bubble of R_a which contains the critical value $-a$. Then $\phi^{-1}(B_a) = W$.*
- (II) *On the other hand, for each bubble $W \in \mathcal{R}$, there exists a unique parabubble P such that for any $a \in P$ we have $\phi(B_a) = W$, where $-a \in B_a$.*
- (III) *Moreover, $a \in \partial P$ if and only if $-a \in \partial B_a$.*
- (IV) *The parabubble P is an open set, has a unique center, and is simply connected.*

Proof. The first and third claim are immediate consequences of Lemma 5.5. The same lemma implies that P is an open set.

We have $\xi_n(a) = R^n(-1, a) \to \partial A_\infty$ as $a \to \partial P$ by Lemma 5.9, so $\Phi_a \circ \xi_n \to \partial \mathbb{D}$ as $a \to \partial P$. By the Argument Principle, this means that every capture component P is mapped by $\Phi_a \circ \xi_n$ onto $\hat{\mathbb{C}} \setminus \mathbb{D}$ as a $d \longrightarrow 1$ covering. We want to show that $d = 1$.

Let P be a parabubble of generation n, and F the corresponding bubble for R_a in which $-a$ lies. Note that the map ϕ in Lemma 4.6 is an injection of all bubbles of generation $\leq n$. Hence we can define $B = \phi^{-1}(F)$. The root of B then is a landing point x of an internal ray of B with angle $\theta = 0$ (by Lemma 2.1). The predecessor C touches B at x. It follows from Lemma 5.7 that an internal parameter ray with angle $\theta = 0$ in P will land at a parameter a such that $R_a^n(-1)$ is the unique repelling fixed point on the boundary of A_∞. It follows that there is a corresponding parabubble Q to C (in the same way as P corresponds to B), such that P touches Q at a. Moreover, $gen(Q) < gen(P)$, since $gen(C) < gen(B)$. Proceeding in this way we see that for every parabubble P, there is a finite sequence of internal parameter rays connecting the center of P with a point on ∂P_∞.

Reversing this process we also see that for every bubble B in the right basilica $\mathcal{R}$ there is a corresponding parabubble P, in the sense that if F is the bubble for R_a in which $-a$ lies, then $B = \phi^{-1}(F)$. We cannot have such correspondence to the left basilica simply because $a = 0$ is a singularity for the family R_a and no sequence of parabubble rays can end there.

We have to prove that there is one and only one bubble in the right basilica corresponding to every parabubble. By Lemma 5.1 the only thing we have to show

is that it is impossible to have one parabubble P corresponding to two different bubbles B_1 and B_2 in the right basilica. This would imply that the parabubble has two distinct centers. By the λ-lemma of [**MSS**], any two centers in the same parabubble P would correspond to quasi-conformally conjugate rational maps. Since these maps would also be postcritically finite, Thurston's Theorem implies that a center is unique. Hence every parabubble corresponds to a unique bubble in the right basilica and (II) is proven.

Now, since the degree of ξ_n coincides with the number of parabubbles of generation n, the Pigeonhole Principle implies that ξ_n has a simple pole at the center of each parabubble of generation n. By the Argument Principle, $\Phi_a \circ \xi_n : P \mapsto \hat{\mathbb{C}} \setminus \mathbb{D}$ assumes every value in $\hat{\mathbb{C}} \setminus \mathbb{D}$ exactly once, so indeed $d = 1$. It follows that every capture component is simply connected.

□

By Lemma 5.9, the mapping

$$\psi : a \mapsto -a \mapsto \phi^{-1}(-a)$$

is an injection from the capture locus of the family R_a to $\overset{\circ}{\mathcal{R}}$. It is straighforward to extend this mapping to the roots of the (para)bubbles, except for the roots contained in the boundary of P_∞.

Denote $\mathcal{C}$ the union of capture components of the family R_a and $\mathcal{C}_e = \mathcal{C} \setminus P_\infty$. Since dynamical bubbles may only touch at a single point, which is a preimage of the fixed point where A_∞ and A_0 as long as $a \in \overline{P}^c_\infty$, our discussion implies:

Proposition 5.10. *If P and Q are two parabubbles not equal to P_∞, and $P' = \psi(P)$, $Q' = \psi(Q)$, then the following holds:*

(1) $\overline{P} \cap \overline{Q} \cap (\overline{P}_\infty)^c = \emptyset \Leftrightarrow \overline{P'} \cap \overline{Q'} = \emptyset$,
(2) $\overline{P} \cap \overline{Q} \cap (\overline{P}_\infty)^c$ *is exactly one point* $\Leftrightarrow \overline{P'} \cap \overline{Q'}$ *is exactly one point,*
(3) $P = Q \Leftrightarrow P' = Q'$.

Moreover,

$$\psi(\mathcal{C}) = \overset{\circ}{\mathcal{R}}.$$

Similarly to the notation for dynamical bubbles, if the intersection of the closures of two parabubbles

$$\overline{P} \cap \overline{Q} = \{a\}$$

is exactly one point and $\mathrm{Gen}(P) > \mathrm{Gen}(Q)$, let us refer to Q as the *predecessor* of P and a as the *root* of P.

Let $\{a_j\}$ be the set of all touching points between parabubbles not including those which lie on the boundary of P_∞. The above proposition implies that ψ continuously extends to a homeomorphism

$$\psi : \mathcal{C}_e \cup \{a_j\} \mapsto \overset{\circ}{\mathcal{R}}_e \bigcup \left(\bigcup_{j=1}^{\infty} (f^{-j}(\alpha) \cap \mathcal{R}_e) \right).$$

Definition 5.3. Let
$$\mathcal{B} = \{F_k\}_0^\infty \subset K_{\circ\!\bigcirc\!\circ}$$
be an infinite bubble ray with angle $\measuredangle(\mathcal{B}) = \theta \in (-1/3, 1/3)$. We call the corresponding sequence of capture components $\{P_k\}_0^\infty$, with $\psi(P_k) = F_k$, a *parabubble ray* in $\mathcal{C}$ with angle θ, and write $\measuredangle(P) = \theta$.

Similarly to the definition for dynamical bubble rays, we define the *axis* for a parabubble ray $\mathcal{P}$ to be the union of the internal parameter rays $\gamma_k, \gamma_k' \subset P_k$ which land at the points $\overline{P}_k \cap \overline{P}_{k-1} = x_k$ and $\overline{P}_{k+1} \cap \overline{P}_k = x_k'$ respectively, starting from ∞.

In the next section we show that certain infinite bubble rays and parabubble rays land at a single point.

6. Landing lemmas

6.1. **Dynamical bubble rays.** We begin with the following lemma.

Lemma 6.1. *Assume that $\mathcal{B}$ is a periodic infinite bubble ray $\mathcal{B}$ such that the axis is disjoint from the closure of the postcritical set. Then the axis γ for $\mathcal{B}$ lands at a single periodic point which is either repelling or parabolic.*

Proof. Let Λ be the closure of the postcritical set and let S be the set of cluster points for γ. If the period of γ to itself is n then R^n maps $\Lambda \cup S$ into itself. Hence R^{-n} can be lifted by the universal covering $\mathbb{D}$ of $\hat{\mathbb{C}} \setminus (\Lambda \cup S)$ to a map $\hat{f} : \mathbb{D} \mapsto \mathbb{D}$ such that $\hat{f}(\mathbb{D}) \subset \mathbb{D}$ is a strict inclusion. Hence R^n is strictly expanding with respect to the Poincaré metric on $\hat{\mathbb{C}} \setminus (\Lambda \cup S)$.

Since γ is invariant under f we can take a starting point $x_0 \in \gamma$ and set $f(x_0) = x_1$, and $x_k = f(x_{k-1})$. Let γ_k be the part of γ between x_k and x_{k+1}. The hyperbolic distance between x_k and x_{k+1} descreases as k increases. Take a point $p \in S$. Then the hyperbolic distance from any point on γ to p is infinite, since S is contained in the boundary of the hyperbolic set $\hat{\mathbb{C}} \setminus (\Lambda \cup S)$. Since the hyperbolic length of γ_k decreases for increasing k, any neighbourhood N of p has the property that there is a smaller neighbourbooh $N' \subset N$ such that if $\gamma_k \cap N' \neq \emptyset$ then $\gamma_k \subset N$. But this means that $f(N) \cap N \neq \emptyset$. So p has to be a fixed point. Since S is connected, S must contain only this point. By the Snail Lemma, p must be a parabolic or repelling point (cf. [**Mi1**], Lemma 16.2). $\square$

We next prove that the axis of a periodic (or preperiodic) bubble ray cannot accumulate on some bubble.

Lemma 6.2. *Let $\mathcal{B}$ be a periodic infinite bubble ray for which the axis is disjoint from the closure of the postcritical set. Then the axis of $\mathcal{B}$ cannot accumulate at some bubble.*

Proof. Without loss of generality, in order to reach a contradiction, it suffices to suppose that the axis of $\mathcal{B}$ accumulates at ∂A_∞. Since $\mathcal{B}$ is periodic, we know from Lemma 6.1 that the axis for the bubble ray $\mathcal{B}$ lands at a single periodic point p on the boundary of A_∞. The bubble ray $\mathcal{B}$ then encloses a domain D whose boundary

is a connected part $I \neq A_\infty$ of ∂A_∞ and half of the boundary of all the other bubbles in $\mathcal{B}$. Since p and $\mathcal{B}$ is fixed under some iterate n we have that D is invariant under R^n. This means that any bubble B in D must never be mapped into $A_0 \cup A_\infty$, since this set lies outside D (the fact that there exists some bubble in D is obvious). This is clearly impossible, since bubbles by definition are preimages of A_∞. □

We are now in position to prove a landing lemma for periodic or preperiodic bubble rays.

Lemma 6.3. *Assume that $\mathcal{B}$ is periodic infinite bubble ray, for which there exist an N such that all bubbles in $\mathcal{B}$ of generation at least N are disjoint from the closure of the postcitical set. Then $\mathcal{B}$ lands at a single point.*

Proof. Assume that $\mathcal{B}$ is periodic of period q. We have seen (Lemma 6.1 and Lemma 6.2) that the axis γ of the bubble ray must land on a periodic point x.

Since the postcritical set Λ is disjoint from any bubble B in $\mathcal{B}$ with $Gen(B) \geq N$, we have an annulus R around this B of some definite modulus $m > 0$ such that there are well defined inverse branches of R_a^{-q} on $R \cup B$, where $R_a^{-qn}(B) \in \mathcal{B}$ for all $n \geq 0$. This means that the lengths of the γ_k in the proof of Lemma 6.1 are commensurable with the diameter of the corresponding bubbles F_k, by the Koebe Distortion Lemma. Hence the bubble ray $\mathcal{B}$ converges to the same periodic point as the axis γ lands on. □

6.2. **Orbit portraits for R_a.** We have seen in Section 4 that bubble rays have angles inherited from the angles of external rays in the basilica (although these angles are not always well defined, as in the capture case for instance). With the theory about orbit portraits for quadratic polynomials in Section 3 in mind, it is now straightforward to define an orbit portrait for R_a.

Definition 6.1. Let $x_1, x_2, \ldots, x_p$ be a (repelling or parabolic) periodic orbit, where $R_a(x_i) = x_{i+1}$, $R_a(x_p) = x_1$. Assume that there are a finite number of periodic infinite bubble rays landing on x_i, with well defined angles; Let A_i be the corresponding angles for the bubble rays landing at x_i. Then *the orbit portrait for R_a* is the set $\mathcal{O} = \{A_1, A_2, \ldots, A_p\}$.

Given two angles $\theta_1 \neq \theta_2$ we let $[\theta_1 \circlearrowleft \theta_2] \subset \mathbb{T}$ be the arc of the unit circle swept by going in counter-clockwise direction from θ_1 to θ_2. We say that θ lies between θ_1 and θ_2 if $\theta \in [\theta_1 \circlearrowleft \theta_2]$.

Before we state the next lemma we make some more definitions.

Definition 6.2. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bubble rays starting from A_∞ with well defined angles θ_1 and θ_2 and axes γ_1 and γ_2. Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ land at a common point p. Denote D the domain bounded by the axes γ_1, γ_2 which does not contain any bubble rays with angles in $\mathbb{T} \setminus [\theta_1 \circlearrowleft \theta_2]$.

Define the *outer boundary* of the sector bounded by $\mathcal{B}_1$, $\mathcal{B}_2$ as the union of the arcs of the boundaries of the bubbles in these two bubble rays lying outside D together with their endpoints. Similarily, define the *inner boundary.* We say that $z \in \hat{\mathbb{C}}$ lies *between* $\mathcal{B}_1$ and $\mathcal{B}_2$ if $z \in D$ and $z \notin \overline{\mathcal{B}}_1 \cup \overline{\mathcal{B}}_2$.

This notion of being *between* two bubble rays also makes sense for bubble rays even if $\mathcal{B}_1$ and $\mathcal{B}_2$ do not land on a common point.

Definition 6.3. Assume that the two bubble rays $\mathcal{B}_1$ and $\mathcal{B}_2$ have intrinsic addresses $s(\mathcal{B}_1) = (x_0, x_1, \ldots,)$ and $s(\mathcal{B}_2) = (y_0, y_1, \ldots)$ respectively. We say that an infinite bubble ray $\mathcal{B}$, with intrinsic address $s(\mathcal{B}) = (z_0, z_1, \ldots)$, lies *between* $\mathcal{B}_1$ and $\mathcal{B}_2$ if $y_i \le z_i \le x_i$ for all $i \ge 0$. Equivalently, the angle

$$\measuredangle(\mathcal{B}) \in [\measuredangle(\mathcal{B}_1) \circlearrowleft \measuredangle(\mathcal{B}_1)].$$

This definition also makes sense for parabubble rays in an exactly analoguous way.

Lemma 6.4. *Let $\mathcal{O} = \{A_1, \ldots, A_p\}$ be a formal orbit portrait with $v_{\mathcal{O}} \ge 2$ and let $I = [t_- \circlearrowleft t_+]$ be its characteristic arc. If the formal orbit portrait $\mathcal{O}$ is realisable by some R_a then $-a$ cannot lie on the outer boundary of a bubble ray with angle t_- or t_+.*

Proof. Since $a \in \mathcal{M}at$, in the case when $-a$ belongs to the boundary of some bubble, we have a conjugacy ϕ from Proposition 4.5 between the dynamics of $f_{\circ\bigcirc\circ}$ on the interior of $K_{\circ\bigcirc\circ}$ and that of R_a on its Fatou set. Now suppose $\mathcal{O}$ is realised and let A_i be the set of angles of the bubble rays landing at x_i.

Assume that A_2 contains the characteristic arc. Let $\mathcal{B}_-$ and $\mathcal{B}_+$ be the bubble rays corresponding to the angles $t_-, t_+ \in A_2$ and let $\mathcal{A}_+, \mathcal{A}_-$ be the bubble rays corresponding to the critical arc in A_1, i.e. so that $\mathcal{A}_-$ and $\mathcal{A}_+$ are mapped onto $\mathcal{B}_-$ and $\mathcal{B}_+$ respectively. Also, let D be the domain enclosed by the axes of $\mathcal{B}_-$ and $\mathcal{B}_+$.

There are two more preimages of $\mathcal{B}_-$ and $\mathcal{B}_+$, call them $\mathcal{A}'_-, \mathcal{A}'_+$ respectively. Also, let $a_- = \measuredangle(\mathcal{A}_-), a_+ = \measuredangle(\mathcal{A}_+)$ and $a'_- = \measuredangle(\mathcal{A}'_-), a'_+ = \measuredangle(\mathcal{A}'_+)$. Since $I_c = (a_-, a_+)$ is the critical arc in A_1 we have that both a'_-, a'_+ lies entirely inside I_c, and thus the bubble rays $\mathcal{A}'_-, \mathcal{A}'_+$ lies entirely inside the domain D_c enclosed by the axes for the bubble rays forming the critical arc.

Now, assume that $-a$ lies on an outer boundary of a bubble in $\mathcal{B}_+$ (the proof is the same if $-a \in \mathcal{B}_-$). Then the critical point -1 must belong to a bubble in $\mathcal{A}_+$. Since R_a is $2-1$ in a neighbourhood of -1 and orientation preserving, we have that the bubble ray $\mathcal{A}'_+$ must touch $\mathcal{A}_+$ at -1. Since $-a$ is outside D, this implies that -1 must be outside D_c. Thus $\mathcal{A}'_+$ must be outside D_c, which is a contradiction. $\square$

The following lemma tells us when a specific orbit portrait is realised.

Lemma 6.5 (**Realization of orbit portraits**)**.** *Let $\mathcal{O} = \{A_1, \ldots, A_p\}$ be a formal orbit portrait with a characteristic arc $\mathcal{I} = [t_- \circlearrowleft t_+]$. Let P_{t_-}, P_{t_+} be the corresponding parabubble bubble rays, with angles t_- and t_+ and assume that a belongs to a parabubble P between P_{t_-} and P_{t_+}. Then the orbit portrait $\mathcal{O}$ is realised by R_a.*

The proof follows that of Lemma 2.9 in [**Mi3**].

Proof. Note that all infinite bubble rays with angles in any A_j are well defined since their forward images do not intersect the critical value.

Let Λ be the closure of the postcritical set for R_a and let $\rho(z)$ be the induced hyperbolic metric on $\hat{\mathbb{C}} \setminus \Lambda$. Let C be the critical bubble containing -1 and V the

critical value bubble containing $-a$. There is a unique finite bubble ray ending at V. Its preimage is two finite bubble rays $\mathcal{B}_1$ and $\mathcal{B}_2$ both ending at C. Their axes γ_1 and γ_2 join in C and form a closed simple curve in $\hat{\mathbb{C}}$.

Take a hyperbolic disk $D \Subset C$ which covers the critical point -1 and let

$$L = \bigcup_{k=0}^{\infty} R_a^k(\gamma_1 \cup \gamma_2 \cup D)$$

It is easy to see that the complement of L is two topological disks U_1 and U_2.

We have $R(L) \subset L$ and $\Lambda \subset L$. Moreover $\mathrm{dist}(\Lambda, L^c) \geq \varepsilon > 0$ for some definite $\varepsilon > 0$. It is easy to check that the nth preimages of U_1 and U_2 consist of 2^{n+1} topological disks.

Moreover, all preimages of the U_j will be on a definite distance $\varepsilon > 0$ from Λ so we have a uniform constant $c = c(\varepsilon) > 1$ so that

$$\rho(R(x), R(y)) \geq c\rho(x, y)$$

for x, y lying in any of these preimages of U_i. It follows that the preimages of U_i shrink to points. Thus the symbol sequence of some point with respect to the initial partition L is unique. In particular the landing points of the periodic bubble rays in $\mathcal{O}$ will have the same symbol sequence if and only if they land at a common point.

To show that $\mathcal{O}$ is indeed realised it now suffices to show that all the landing points of the bubble rays with angles in A_j lie entirely in one of the components U_i. Since they are mapped onto each other they will have the same symbol sequence in that case.

The preimages of the characteristic arc $[t_{-1} \circlearrowleft t_+]$ under the doubling map will be two smaller arcs I' and I'' at the end of the critical arc. Since every $A_j \in \mathcal{O}$ cannot have any element in I' or I'' we have that all bubble rays corresponding to angles in A_j are completely contained in U_1 or completely contained in U_2. Thus all the angles in every A_j have the same symbol sequence, so they land on a common point, and so $\mathcal{O}$ is a realised bubble portrait. $\square$

Lemma 6.6. *Assume that R_a has a parabolic fixed point z_0, with $R_a'(z_0) = e^{2\pi i p/q}$, where $p/q \in \mathbb{Q}$ with $(p, q) = 1$. Then there are precisely q periodic bubble rays $\mathcal{B}_j$, $j = 1, \ldots, q$, landing at z_0. These bubble rays are mapped onto each other under the action of R_a, with combinatorial rotation number p/q.*

Proof. For simplicity, consider the mapping R_a with $a = 32/27$ which has a simple parabolic with eigenvalue 1. After a suitable change of coordinates shifting the fixed point to the origin, this mapping takes the form

$$\zeta \mapsto \zeta + \zeta^2 + \mathcal{O}(\zeta^3)$$

in a neighborhood of $\zeta = 0$. Denote $\mathcal{A}$ and $\mathcal{R}$ the attracting and repelling petals of R_a correspondingly. Note that Montel's Theorem guarantees that the repelling petal contains a bubble B.

Now, B is the end of some finite bubble ray $\mathcal{C}_F$. Taking the preimages of the bubble ray $\mathcal{C}_F$ we get a sequence of bubble rays $\mathcal{C}_k = R_a^{-k}(\mathcal{C}_F)$, whose ends will converge to z_0.

Since preimages will increase the generation and since there are finitely many finite bubble rays of any fixed generation, for any N there must be some bubble $\mathcal{B}_0$ of generation N contained in infinitely many $\mathcal{C}_k$. Let $\mathcal{C}_{k_0} \supset \mathcal{B}_0$ be the $\mathcal{C}_k$ containing $\mathcal{B}_0$ with lowest generation and $\mathcal{C}_{k_1} \supset \mathcal{B}_0$ the second lowest. Then $R_a^m(\mathcal{C}_{k_1}) = \mathcal{C}_{k_0}$ for some $m \geq 1$ and the preimage of $\mathcal{B}_0$ under R_a^m is a longer bubble ray $\mathcal{B}_1 \supset \mathcal{B}_0$. Moreover, $\mathcal{B}_1 \subset \mathcal{C}_{k_1}$. Taking further preimages of $\mathcal{B}_1$ under the same branch $f = R_a^{-m}$ we get a sequence $\mathcal{B}_n$ of nested finite bubble rays such that $\mathcal{B}_n \subset \mathcal{C}_{k_n}$. Moreover, the "difference" between $\mathcal{B}_n$ and $\mathcal{C}_{k_n}$, i.e. the number of bubbles in $\mathcal{D}_n = \mathcal{C}_{k_n} \setminus \mathcal{B}_n$ is a fixed constant K for all n. The bubble $\mathcal{D}_n$ is also a preimage of the starting set $\mathcal{D}_0$ under f. Since the postcritical set Λ accumulates on z_0, it is disjoint from $\mathcal{D}_n$. Thus there is a neighbourhood around all bubbles in $\mathcal{D}_0$ where f^n is defined for all $n \geq 0$. Now, the Koebe Distortion Lemma implies that all bubbles in $\mathcal{D}_n$ shrinks to points, namely the parabolic fixed point z_0, since one of them, namely the end of $\mathcal{C}_{k_n} \supset \mathcal{D}_n$, converges to z_0. Hence there is a subsequence of bubbles in $\mathcal{B}_n$ which converge to z_0 (but we do not know a priori that the bubble ray itself will converge to z_0).

However, by construction, the bubble ray $\mathcal{B} = \cup_n \mathcal{B}_n$ is periodic. We can now apply Lemma 6.3 to $\mathcal{B}$, which shows that $\mathcal{B}$ lands at a single point, which must be equal to z_0.

Let us show that the period of $\mathcal{B}$ is 1. A priori, $\mathcal{C}$ is periodic with a period which divides m. Assume the period is $p \neq 1$ and that $R_a(\mathcal{B}_j) = \mathcal{B}_{j+1}$ for $1 \leq j \leq p-1$, $R_a(\mathcal{B}_p) = \mathcal{B}_1$. By simple combinatorial considerations (see e.g. [**Mi3**]), these bubbles form their own orbit portrait. But this means that some point $z \in \mathcal{R} \setminus \mathcal{A}$ in the domain bounded by two consecutive bubble rays $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$, will be mapped into $\mathcal{A}$, which is impossible. Hence $p = 1$. □

6.3. **Parameter bubble rays.**
Let us first note the following evident statement:

Lemma 6.7. *Assume that an orbit portrait $\mathcal{O}$ is realized for some rational map R_a by bubble rays landing at a repelling orbit $\{x_i\}$. Let a_t, $t \in [0,1]$ be a continuous path with $a_0 = a$ along which the corresponding periodic orbit $\{x_i^t\}$ remains repelling. Assume further that for every t no iterate of the critical value $-a_t$ is contained in the boundary of a bubble ray with angle $\gamma \in \mathcal{O}$. Then the orbit portrait $\mathcal{O}$ is realized for all R_{a_t}.*

The following Proposition has an analogue in [**Mi1**], Theorem 4.1 (and Lemma 4.2). Since the proof is completely similar, we omit it.

Proposition 6.8 (Milnor; Parameter Path)**.** *Given a parameter a_0 such that R_{a_0} has a parabolic fixed point z_0 with combinatorial rotation number p/q and an orbit portrait $\mathcal{O}$ (from Lemma 6.6). Then there is a path γ emerging from a_0 in parameter space so that $a \in \gamma$ implies that R_a has a repelling fixed point $z = z(a)$ with orbit portrait $\mathcal{O}$ and an attracting periodic orbit with period q, close to $z(a)$.*

The set A of parameters where the attracting periodic orbit in the above lemma exists, is bounded by a finite number of analytic curves. Indeed,

$$A = \{a : |(R^q)'(z_i(a), a)| < 1\}.$$

The condition $|(R^q)'(z_i(a), a)| = 1$ represents an analytic curve with a finite number of singularities. We conclude that there is a "wedge" $\tilde{W}$, that is, two analytic curves γ_1 and γ_2 which meet at a_0 such that for a small neighbourhood $B(a_0, \varepsilon)$, an open set E bounded by γ_1, γ_2 and $\partial B(a_0, \varepsilon)$ has the property that inside E, we have $\mathcal{O}$ realised and $z_i(a)$ is an attracting periodic orbit of period q (as in the above lemma).

By Lemma 6.7 and Lemma 6.4 the parabubble rays $\mathcal{P}_{t^+}, \mathcal{P}_{t^-}$ lie outside of the wedge $\tilde{W}$.

Lemma 6.9. *Let a_0 be as in the above lemma and assume that (t^+, t^-) is the characteristic arc for $\mathcal{O}$. Then for any $\varepsilon > 0$, we have $B(a_0, \varepsilon) \cap \mathcal{P}_t \neq \emptyset$, for at least one $t = t^+, t^-$, where $\mathcal{P}_{t^+}, \mathcal{P}_{t^-}$ denote the parabubble rays with angles t^+, t^- respectively.*

Proof. Assume the contrary. Then there is an $\varepsilon > 0$ such that $B(a_0, \varepsilon)$ is disjoint from the parabubble rays $\mathcal{P}_t$ for $t = t^+, t = t^-$. By the above argument, and Theorem 6.8, the orbit portrait $\mathcal{O}$ is realised in $B(a_0, \varepsilon) \cap N$, where $N = \{a : |R'_a(\alpha(a))| > 1\}$, and $\alpha(a)$ is the (local) continuation of the parabolic fixed point z_0 (this is possible if the multiplier is $\neq 1$). Hence there is a parameter $a_1 \in B(a_0, \varepsilon)$, such that R_{a_1} also has a parabolic fixed point z_1.

But since the combinatorial rotation number is changed for a_1 the new wedge $\tilde{W}_1$ emerging from a_1 has to exhibit a different orbit portrait $\mathcal{O}_1$. But $\tilde{W}_1$ must intersect $B(a_0, \varepsilon) \cap N$, and so both orbit portraits $\mathcal{O}_1$ and $\mathcal{O}$ are realised, which is impossible. The lemma follows. $\square$

Proposition 6.10 (Parabubble wakes I). *Let a_0 be such that R_{a_0} has a parabolic fixed point z_0 with eigenvalue $R'_{a_0}(z_0) = e^{2\pi i p/q}$, $(p, q) = 1$. Denote $\mathcal{O} = \{\{\theta_1, \ldots, \theta_q\}\}$ the orbit portrait from Lemma 6.6, and let $\mathcal{I} = [t_- \circlearrowleft t_+]$ be its characteristic arc. Then the corresponding parabubble rays with angles t_+ and t_- land on a_0.*

Proof. The standard considerations of parabolic dynamics imply that

$$R^q_{a_0}(z) = R(z) = (z - z_0) + b(z - z_0)^{q+1} + \mathcal{O}((z - z_0)^{q+2}),$$

for some $b \neq 0$. For a close to a_0 the fixed point z_0 will bifurcate into $q+1$ fixed points (for R^q_a) $z_k(a)$, which are analytic in a neighbourhood of a_0, and where $z_k(a_0) = z_0$, for $k = 1, \ldots, q+1$. One of these fixed points must be a fixed point for R_a as well if $q \geq 2$, while the other fixed points (for R^q_a) are all repelling, indifferent or attracting. By Lemma 6.9 there must be a subsequence of parabubbles $P_{n_k} \subset \mathcal{P}_{t^+}$ (or $\mathcal{P}_{t^-}$) such that $P_{n_k} \cap B(a_0, \varepsilon) \neq \emptyset$, for all $k \geq N(\varepsilon)$. Hence, for sufficiently large k, if $a_1 \in P_{n_k}$, then $-a_1 \in B^{n_k}$, where B^{n_k} is the corresponding dynamical bubble in the bubble ray $\mathcal{B}_{t^+}$, i.e. with same address as P^{n_k}. Since a_1 is a capture parameter the fixed points $z_i(a_1)$ (under R^q_a) cannot be attracting. They cannot be neutral so they must be repelling.

We now use the standard theory of parabolic bifurcation (see for ex [**Sh2**] Section 7, [**Sh3**], [**DH1**]). For a suitable small perturbation, we get q fundamental domains $S^k_{+,a}$ and $S^k_{-,a}$, $1 \leq k \leq q$, for the repelling and attracting petals respectively for the

perturbed map R_a. They have the property that

$$S^k_{+,a} \cap S^k_{-,a} = \{\alpha(a), z_k(a)\}.$$

Moreover, there exist analytic functions $\Phi^k_{+,a}, \Phi^k_{-,a}$ (the perturbed Fatou coordinates) which are defined and injective in a neighbourhood of $\tilde{S}^k_{+,a} = S^k_{+,a} \setminus \{\alpha(a), z_k(a)\}$ and $\tilde{S}^k_{-,a} = S^k_{-,a} \setminus \{\alpha(a), z_k(a)\}$ respectively, and conjugate the dynamics of R^q_a to that of the unit translation. With a choice of normalization, these coordinates will vary locally analytically with a.

If $z \in \tilde{S}^k_{-,a}$, then there is an $n \geq 1$ such that $R^{qn}_a(z) \in \tilde{S}^k_{+,a}$, and for the smallest such n,

$$\Phi^k_{+,a}(R^{qn}_a(z)) = \Phi^k_{-,a}(z) - \frac{1}{\beta(a)} + n + \text{const},$$

where $\beta(a) = \beta_k(a)$ is an analytic function in a punctured neighbourhood of a_0, $\beta(0) = 0$, defined by

$$(R^q_a)'(\alpha(a)) = e^{2\pi i \beta(a)}.$$

Denote C^k_+, C^k_- the Écalle-Voronin cylinders, obtained as the quotients

$$C^k_+ = S^k_{+,a} \operatorname{mod} R^q_a \simeq \mathbb{C}/\mathbb{Z},\ C^k_- = S^k_{-,a} \operatorname{mod} R^q_a \simeq \mathbb{C}/\mathbb{Z}.$$

We get that for $z \in C^k_+$,

$$\Phi^k_{-,a} \circ R^{qn}_a \circ (\Phi^k_{+,a})^{-1}(z) = z + \frac{1}{\beta(a)} \operatorname{mod} \mathbb{Z}. \tag{6.1}$$

The function

$$\tau_a(z) = \frac{1}{\beta(a)} + z \operatorname{mod} \mathbb{Z}$$

viewed as an isomorphism $C^k_+ \mapsto C^k_-$ is called the transit map.

Now let us fix some k so that the critical point -1 belongs the the kth attracting petal. For simplicity let us drop the indices k in the above discussion and only focus on these particular Fatou coodinates. Then for any prescribed bubble B_l in $\mathcal{B}_{t^+}$ we can find a parameter $a \in B(a_0, \varepsilon)$ such that $R^{qn}_a(-1) \in B_l$, for some $n \geq 1$, $n = n(l)$.

Fix $a = a_1$ as above. For this specific perturbation, we already have $-a \in B^{n_k}$, so we know that $n = 1$ in (6.1). The bubbles B_n move holomorphically and with uniformly bounded distortion in the Fatou coordinates for a in some disk $B(a_0, \varepsilon)$ (the lifted dynamical bubbles in $\mathcal{B}_{t^+}$, in the Fatou coordinates, are all unit translates of each other). The function $\tau_a(-1) = 1/\beta(a) + z \operatorname{mod} \mathbb{Z}$ has derivative

$$\partial_a \tau_a(z) \sim D \frac{1}{(a - a_0)^m} = \frac{-m}{(a - a_0)^{m+1}},$$

for some $m \in \mathbb{Q}$, $m > 0$. This, and the distortion considerations, imply that P_{n_k} converge to a_0.

It remains to show that *all* parabubbles $P_l \subset \mathcal{P}_{t^+}$ converge to a_0, instead of just a subsequence l_k. This follows from the fact that $n = n(a)$ in (6.1) is continuous function of a which only assumes integer values. Hence $n = 1$ for all l and $\mathcal{P}_{t^+}$ lands on a_0. Of course a similar statement holds for $\mathcal{P}_{t^-}$.

□

Let us write $W = W(t^+, t^-)$ for the *parabubble wake* being set of points between the parabubble rays from the above lemma. Also, let $\mathcal{O} = \mathcal{O}(t^+, t^-)$, be the corresponding orbit portrait. Note that the characteristic arcs corrsponding to different orbit portraits around the fixed point are disjoint.

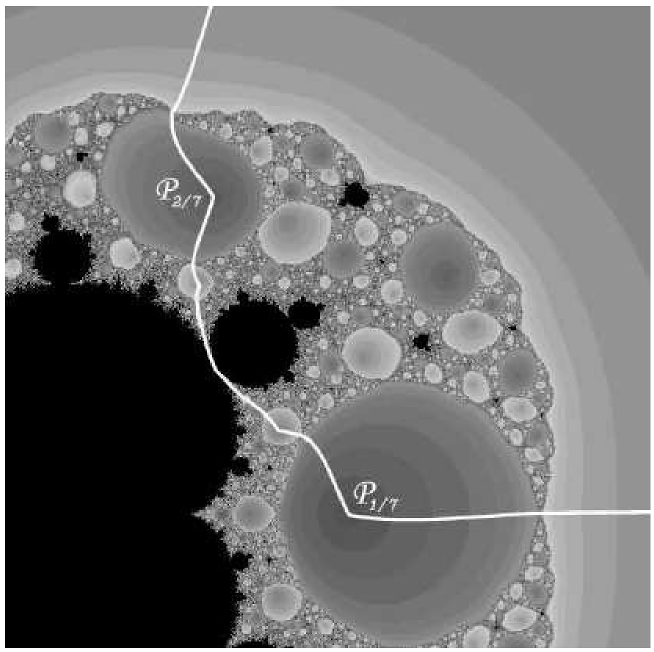

FIGURE 5. An example of a parameter wake, $W(1/7, 2/7)$. The axes of the parabubble rays $\mathcal{P}_{1/7}, \mathcal{P}_{2/7}$ which bound the wake are indicated. Their common landing point is the parameter value a_0 for which R_{a_0} has a parabolic fixed point z_0 with eigenvalue $e^{2\pi i/3}$. The orbit portrait of z_0 is $\{1/7, 2/7, 4/7\}$.

Lemma 6.11 (Parabubble wakes II). *The parabubble rays in the above lemma cut out an open set in the complex plane, called the* bubble wake $W = W(t^+, t^-)$ *such that $a \in W$ if and only if R_a exhibits the repelling orbit portrait $\mathcal{O} = \mathcal{O}(t^+, t^-)$.*

Proof. By Lemma 5.9 the set A_∞ moves holomorphically and the critical value $-a$ belongs to the boundary of a bubble if and only if a belongs to the boundary of the corresponding parabubble. By Lemma 6.7 if for a single parameter $a \in W = W(t^+, t^-)$ the map R_a realises the orbit portrait $\mathcal{O} = \mathcal{O}(t^+, t^-)$, then the same is true for every parameter in W.

On the other hand, $\mathcal{O}$ cannot be realised for any parameter value outside W. Indeed, $\mathcal{O}$ is not realised for a in any of the capture components outside W, since this would imply that the critical value is outside the characterisic arc. □

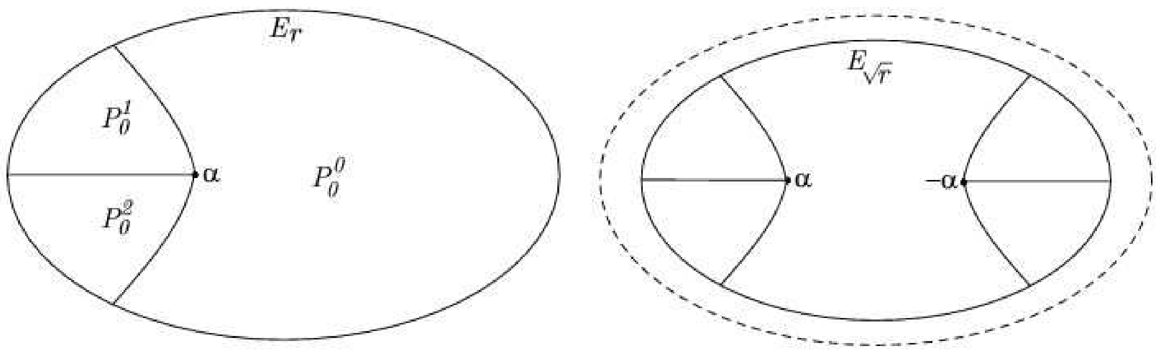

FIGURE 6. The Yoccoz puzzles of depths 0 and 1, with $q = 3$ external rays landing at α.

7. A PUZZLE PARTITION FOR R_a

The idea of a puzzle partition for a Julia set originated in the work of Branner and Hubbard [**BH**]. It has been further developed by Yoccoz (see e.g. [**Hub**] and [**Mi5**]), to study the local connectedness of the Mandelbrot set at Yoccoz parameters, and the local connectedness of the corresponding Julia sets. We employ the Branner-Hubbard-Yoccoz approach to maps of the family R_a using partitions given by landing bubble rays.

7.1. **The Yoccoz puzzle for quadratic polynomials.** Let us recall the main steps of Yoccoz' construction for a quadratic polynomial f_c without non-repelling orbits with a connected Julia set. Let α stand for the dividing fixed point of f_c. It is the landing point of a cycle of $q > 1$ external rays of f_c. Denote these rays $R_1, \ldots, R_q$. Recall that the Böttcher coordinate

$$\Phi : \hat{\mathbb{C}} \setminus K(f_c) \mapsto \hat{\mathbb{C}} \setminus \mathbb{D},$$

conjugates f_c to the dynamics of $z \mapsto z^2$. Fix an arbitrary $r > 1$ and let E_r be the equipotential curve

$$E_r = \Phi^{-1}(\{re^{2\pi i\theta} : \theta \in [0, 1]\}.$$

Let U_0 be the graph formed by

$$U_0 = R_1 \cup \cdots \cup R_q \cup E_r \cup \{\alpha\}.$$

The *puzzle pieces of depth* 0 are the bounded components of $\mathbb{C} \setminus U_0$. Denote these q topological disks P_0^j, $j = 0, \ldots q - 1$. By definition, the Yoccoz' puzzle pieces of depth $d \geq 1$ are the first preimages of the puzzle pieces of depth $d - 1$ under f_c.

What makes puzzle partitions of Julia sets so useful in the study of local connectedness are the following two straightforward observations:

Proposition 7.1. *The following two properties hold:*

- *(Markov property) any two puzzle pieces* P_d^j *and* $P_{d'}^{j'}$ *are either disjoint, or one of them is contained in the other;*
- *the intersection* $J_c \cap P_d^j$ *is a connected set.*

The Markov property allows us to make the following definition for any point $z \in J_c$ which is not a preimage of α.

Definition 7.1. For any $z \in J_c$ with $\alpha \notin \cup f_c^n(z)$, let $P_d(z)$ denote the puzzle piece of depth d which contains z. Let us also set

$$A_d(z) = P_d(z) \setminus \overline{P_{d+1}(z)}.$$

We will refer to $A_d(z)$ as an annulus, even though it may be degenerate. The sequence of annuli $A_d(0)$ will be called the *critical annuli*.

The following is a consequence of Grötzch Inequality (see e.g. [**BH**]):

Lemma 7.2. *Let* A_i*,* $i \in \mathbb{N}$ *be a sequence of bounded conformal annuli in the plane with simply-connected complementary components. Denote* W_i *the bounded component of* $\mathbb{C} \setminus \bar{A}_i$*. Assume that* $A_{i+1} \subset W_i$ *and*

$$\sum \operatorname{mod} A_i = \infty.$$

Then

$$\operatorname{diam}\left(\bigcap W_i\right) = 0$$

Yoccoz has demonstrated, in particular:

Lemma 7.3. *Assume that* f_c *is non-renormalizable. Then*

$$\sum \operatorname{mod} A_d(0) = \infty.$$

His proof uses the concept of a *tableau* developed by Branner and Hubbard [**BH**]. Below we extract a definition suitable for a generalization from [**Mi5**]. To motivate some of the notation, fix a point $z \in J_c$, and consider its orbit under f_c:

$$z = z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots.$$

Note that the puzzle piece $P_d(z_j)$ is mapped onto $P_{d-1}(z_{j+1})$, either as a conformal isomorphism or a branched double covering, depending on whether the piece $P_d(z_i)$ contains the critical point or not.

Definition 7.2. Let $S(z)$ be the largest integer $d \geq 0$, for which $P_d(z) = P_d(0)$. If $P_d(z) = P_d(0)$ for all d, put $S(z) = \infty$, and if $P_d(z) \neq P_d(0)$ for all d, put $S(z) = -1$.

We then distinguish the following three possibilities:

- **Critical case:** $d < S(z_i)$. Here the critical point lies in $P_d(z_i) = P_d(0)$. Hence the annulus $A_d(z_i)$ is mapped onto its image as am unbranched two-to-one covering. One easily deduces that

$$\operatorname{mod} A_d(z_i) = \frac{1}{2} \operatorname{mod} A_{d-1}(z_{i+1}).$$

- **Off-critical case:** $d > S(z_i)$. Here the critical point is outside $A_d(z_i)$ so that $A_d(z_i)$ is mapped conformally onto its image $A_{d-1}(z_{i+1})$. Indeed,
$$\operatorname{mod} A_d(z_i) = \operatorname{mod} A_{d-1}(z_{i+1}).$$
- **Semi-critical case:** $d = S(z_i)$. This means that the critical point lies in the annulus $A_d(z_i)$, and
$$\operatorname{mod} A_d(z_i) > \frac{1}{2} \operatorname{mod} A_{d-1}(z_{i+1}).$$

Definition 7.3 (**A critical tableau).** A critical tableau is a two-dimensional array of non-negative real numbers $(\mu_{d,n})$, $d, n \geq 0$ together with a marking, formed according to a set of rules given below. Each position of the tableau is marked as *critical*, *semi-critical*, or *off-critical*. An *iterate* $\mathcal{I}$ in the tableau is a move in the north-western direction in the array:
$$\mu_{d,n} \underset{\mathcal{I}}{\longrightarrow} \mu_{d-1,n+1}.$$
The rules of a critical tableau are as follows.

- Every column of a tableau is either all critical; or all off-critical; or has exactly one semi-critical position (d_0, n) and is critical above $(d > d_0)$ and off-critical below. The 0-th column is all critical.
- If
$$\mu_{d,n} > 0 \text{ then } \mathcal{I}(\mu_{d,n}) > 0.$$
Moreover, if (d, n) is marked off-critical, then $\mathcal{I}(\mu_{d,n}) = \mu_{d,n}$;
if (d, n) is marked semi-critical, then $\mathcal{I}(\mu_{d,n}) < 2\mu_{d,n}$;
if (d, n) is marked critical, then $\mathcal{I}(\mu_{d,n}) = 2\mu_{d,n}$.
- Let position (d_0, n) be marked as either critical or semi-critical. Draw a line north-east from this position, and do the same from the position $(d_0, 0)$ in the tableau. Then the marking above the second line must be copied above the first one.
- Suppose that $(d, 0)$ is marked critical, $(d - k, k)$ is also critical, and $(d - i, i)$ is off-critical for $i < k$. Assume that (d, n) is semi-critical for some n. Then $(d - k, n + k)$ is also semi-critical.

Finally, we say that a tableau is *recurrent* if
$$\sup\{d|\ (d, k) \text{ is critical for some } k > 0\} = \infty;$$
we say that it is *periodic* if there exists $k > 0$ such that the k-th column is entirely critical.

The relevance to the quadratic Yoccoz' puzzle should be evident from the above discussion:

Definition 7.4 (**The critical tableau of a Yoccoz' puzzle).** For f_c as above, we let
$$\mu_{d,n} = \operatorname{mod} A_d(f_c^n(0)).$$

We note:

Proposition 7.4. *The critical tableau of the Yoccoz' puzzle of f_c is periodic if and only if f_c is renormalizable.*

The basis of the Yoccoz' result is given by the following theorem:

Theorem 7.1. *Assume that $(\mu_{d,n})$ is a tableau, which is recurrent and not periodic. Assume further that there exists d such that $\mu_{d,0} > 0$. Then*

$$\sum_d \mu_{d,0} = \infty.$$

7.2. **A puzzle partition for R_a.** The puzzle pieces for R_a which we construct are similar to those just described but instead of external rays we use bubble rays. More specifically, choose a parameter a in a parabubble wake $W(t^+, t^-)$, and let the corresponding orbit portrait be

$$\mathcal{O}(t^+, t^-) = \{\{\theta_1, \ldots, \theta_q\}\}.$$

Denote $\mathcal{B}_i = \mathcal{B}_{\theta_i}$ the bubble ray with angle θ_i starting with the bubble A_∞, and let α_a be the common landing point of these rays. Another repelling fixed point of R_a, that in the intersection of $\bar{A}_0$ and $\bar{A}_\infty$ will be denoted p_a.

Definition 7.5. The *thin initial puzzle-pieces* of R_a are the connected components of

$$\hat{\mathbb{C}} \setminus \left(\overline{(\cup_i \mathcal{B}_i)} \cup \{\alpha_a\} \right).$$

Similarly, a *thick initial puzzle-piece* of R_a corresponding to a thin puzzle-piece P is the set

$$\bar{P} \cup \mathcal{B}^1 \cup \mathcal{B}^2,$$

where $\mathcal{B}^i$ are the two bubble rays which bound P.

Finally, an *initial puzzle-piece* of R_a is a domain obtained as follows. Let γ_i be the axis of $\mathcal{B}_i$ terminating at α and ∞. Further, let

$$\Phi : A_\infty \mapsto \hat{\mathbb{C}} \setminus \mathbb{D}$$

be the Böttcher coordinate, fix an arbitrary $r > 1$, and let

$$D = \Phi^{-1}(\{|z| > r\}) \text{ and } D' = R_a^{-1}(D_r) \cap A_0.$$

The initial puzzle-pieces are the connected components of

$$\hat{\mathbb{C}} \setminus \left((\cup \gamma_i) \cup \{\alpha_a\} \cup \bar{D} \cup \bar{D}' \right).$$

We denote the initial puzzle-pieces $P_0^1, \ldots, P_0^q$. The puzzle pieces of *depth* n are the n-th preimages of P_0^i, they will be denoted P_n^j.

The basic properties being the same for all three kinds of puzzle-pieces, we will only formulate the results for the last kind. We begin by noting:

Lemma 7.5 (**Markov property**)**.** *For any two puzzle pieces P_n^i, P_m^j one of the following two possibilities holds: they are disjoint, or one is a subset of the other.*

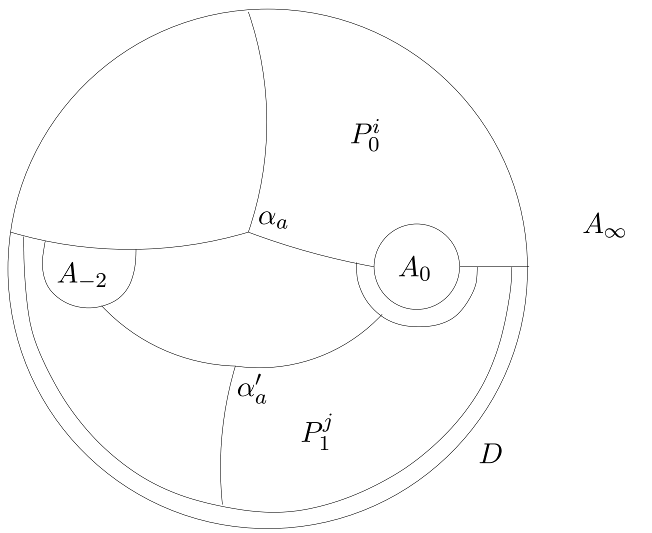

FIGURE 7. A bubble puzzle of depth 1. Note that the pieces P_0^i and P_1^j touch at an arc connecting A_0 and A_∞.

This allows us again to define for a point $z \in J(R_a)$ which is not a preimage of α_a $P_d(z)$ as the puzzle-piece of depth d which contains z. Further, set

$$A_d(z) = P_d(z) \setminus \overline{P_{d+1}(z)};$$

we refer to this set as a complementary annulus, although it could be degenerate. We again label the annuli as critical, off-critical, and semi-critical depending on the position of the critical point -1. A critical annulus $A_{d+k}(-1)$ will be called a *child* of the critical annulus $A_d(-1)$ if

$$R_a^k : A_{d+k}(-1) \to A_d(-1)$$

is an unramified double covering.

We define $\mathcal{T}_a$ to be a marked array

$$\mathcal{T}_a = (\operatorname{mod} A_d(R_a^n(-1))),\ d, n \geq 0,$$

with the positions marked as critical, off-critical, or semi-critical if the respective annuli are. The following Proposition is verified in a straightforward way, completely similarly to the quadratic case. We therefore omit the proof.

Proposition 7.6. *The marked array $\mathcal{T}_a$ is a critical tableau.*

However, it may happen that there is no non-degenerate annulus in the tableau $\mathcal{T}_a$. We will need to modify the construction of the annuli slightly to guarantee the existence of one.

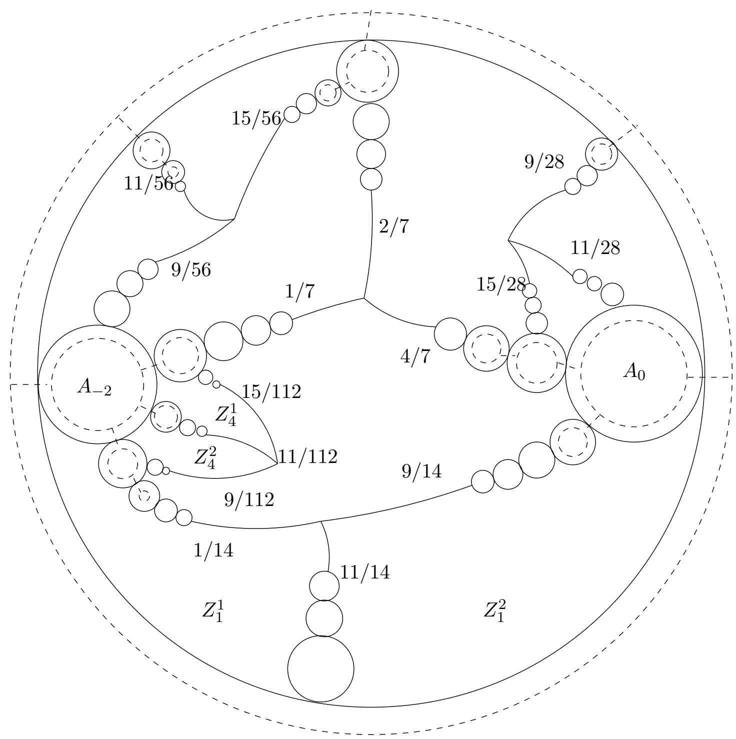

FIGURE 8. A bubble-puzzle of depth 1 together with some preimages of the pieces Z_1^1 and Z_1^2. The broken lines show the "equipotential" of depth 4. Note that Z_4^1 is degenerate in the sense that its boundary touches the boundary of $P_0(-1)$, whereas Z_4^2 is not.

7.3. **Non-degenerate annuli.** The construction of a non-degenerate critical annulus for R_a is somewhat more delicate than that for a quadratic polynomial. We begin with the following:

Lemma 7.7. *We have $P_1(-1) \Subset \hat{\mathbb{C}} \setminus \bar{D}$.*

Proof. There are $q \geq 3$ infinite bubble rays $\mathcal{B}_k$, $k = 1, \dots q$ landing at α. First, let us argue that at least one bubble ray $\mathcal{B}_k$ contains A_{-2} (the Fatou component of R_a containing -2) and another contains A_0. Suppose this is not the case. Then all, but possibly one, external angles θ_k for B_k will belong to $(1/6, 1/3) \cup (2/3, 5/6)$. But then all, but possibly one, of the images of θ_k under doubling will belong to $(1/3, 2/3)$, which is disjoint from $(1/6, 1/3) \cup (2/3, 5/6)$. Since $q \geq 3$ this gives a contradiction.

We want to show that the preimages $\mathcal{B}'_k$ of $\mathcal{B}_k$ landing at the preimage of α have the same property, that is at least one bubble ray $\mathcal{B}'_k$ contains A_{-2} and another contains A_0. If this is not the case then the images of all, but possibly one, $\mathcal{B}'_k$ have angles in $(1/3, 2/3)$, which is impossible.

Hence the region $P_1(-1)$ is bounded by four bubble rays which all emerge from A_{-2} or A_0. It is easy to see that this region is compactly contained in A^c_∞, and the lemma follows. □

Now let us denote $Z^1_1, \dots, Z^1_{q-1}$ the puzzle-pieces of level 1 which are not adjacent to α_a, but to its other preimage α'_a. It is easy to see that if $\mathcal{T}_a$ is not a periodic tableau, then some iterate of the critical point -1 under R^q_a will escape to one of the pieces Z^j_1. The first time this happens, say after the n-th iterate, we can pull back the *degenerate* annulus $P_0(-1) \setminus P^j_1$ under R^{qn}_a. See Figure 8 for an illustration. This will give a degenerate critical annulus $A_m(-1)$. However, by Lemma 7.7, the only place where the boundaries of $P_m(-1)$ and $P_{m+1}(-1)$ touch is a preimage of the segment l of two internal rays containing $\overline{A_\infty} \cap \overline{A_0}$ which connects D and D'. The invariance of $A_\infty \cup A_0$ implies:

Lemma 7.8. *The pinching of any child of $A_m(-1)$ is disjoint from $\overline{P_m(-1)} \cap \overline{P_{m+1}(-1)}$.*

This means in particular the following:

Corollary 7.9. *Let $A_m(-1)$ be as above. Let $A_{m_j}(-1)$ be any child of $A_m(-1)$. Then the critical puzzle pieces $P_{m_j}(-1)$ satisfy*

$$P_{m_{j+1}}(-1) \Subset P_{m_j}(-1) \text{ and } P_{m_{j+1}+1}(-1) \Subset P_{m_j+1}(-1).$$

We first handle the non-recurrent case:

Lemma 7.10. *If there is some N so that $P_N(-1)$ is disjoint from the orbit $z_0 \mapsto z_1 \mapsto \dots$, then $\cap_n P_n(z_0) = \{z_0\}$.*

Proof. The proof goes as in [**Mi5**]. We first thicken the puzzle-pieces of level $N-1$ to domains $U_i \supset P^i_{N-1}$, numbered so that $U_0 \supset P_{N-1}(-a)$ and with the following property: for each $i > 0$ there are two univalent branches g^i_1 and g^i_2 of R^{-1}_a defined on U_i, each of which carries it into a proper subset of some U_j. This is easily done, we leave the details to the reader. We next equip every U_i with the Poincaré distance $\rho_i(x, y)$. It follows that for each puzzle piece P^i_{N-1}, $i > 0$, the branch g^i_k shrinks the Poincaré distance by some definite factor $\lambda < 1$. Since the orbit $z_0, z_1, z_2, \dots$ avoids the critical puzzle piece we get that

$$\operatorname{diam}(P_{N+h}(z_0)) \leq \delta\lambda^h,$$

and the statement of the lemma follows.

□

We next attack the harder recurrent case:

Theorem 7.2. [1]

Assume that the critical tableau $\mathcal{T}_a$ is recurrent and not periodic. Then

$$\bigcap P_d(-1) = \{-1\}.$$

[1]We thank Carsten Petersen for pointing out that the proof of Theorem 7.2 given in the published version may fail in some cases. We supply the corrected proof below.

Assume A is a degenerate critical annulus. We may assume that A is excellent. Hence every child is excellent as well. This forms a tree of descendants $A_{i,j}$ starting from $A = A_{0,1}$ so that, for fixed $i > 0$, $A_{i,j}$ are the descendants of generation i. Generation i means that $f^i(A_{i,j}) = A_0$ and that $f^k : A_{i,j} \to A_0$ is a 2^i degree unbranched covering. Moreover, since every $A_{i,j}$ is excellent there are at least 2^i annuli of generation i.

All $A_{i,j}$ form a nest around the critical point. We can relabel them so that $A = A_0$ surrounds A_1 which in turn surrounds A_2 and so on. In this way we get a nested sequence of annuli.

The *complementary annulus* α_j is defined to be the annulus between the outer boundary of A_j and the outer boundary of A_{j-1}. Note that by Corollary 7.9 any complementary annulus is non-degenerate.

Theorem 7.2 will follow from:

Lemma 7.11. *The sum of the moduli of all complementary annuli is infinite.*

An annulus is a difference between two puzzles pieces, an outer puzzle piece and an inner puzzle piece, the inner being contained in the outer. We say that an annulus A *surrounds* a set E if the inner puzzle piece of A contains E.

Take some complementary α which lies between the two degenerate annuli $P = A_l$ and $Q = A_{l+1}$, where P surrounds Q. Note that we assume that no annulus A_j lies strictly between P and Q. Now Q has a child, say Q_1, so that Q_1 maps onto Q as a 2 degree unbranched covering. We want to pull back P along the same branch (if possible) as Q back to some P_j surrounding Q_1.

In the first steps α (between P and Q) is pulled back as a one-to-one map until some preimage P_1 of P under f^k surrounds the critical point. This means by definition that this preimage P_1 is a child to P. If, moreover, Q_1, being the preimage of Q under f^k surrounded by P_1, also surrounds the critical point, then we stop and have found P_1 surrounding Q_1 both being children of P and Q respectively. Since we assumed that no degenerate annulus A_j is between P and Q, it follows that there cannot be any such A_i between P_1 and Q_1 either.

The second, and more likely, case is that, whereas P_1 surrounds the critical point, Q_1 does not surround the critical point. Hence we are in a semi-critical situation, so the pullback $f^{-k}(\alpha)$ is not an annulus. However, if we consider the annulus β_1 between P_1 and Q_1, this annulus has modulus at least $1/2$ of the modulus of α, by a standard inspection from semi-critical annuli. Continuing pulling back β_1, we again sooner or less reach the same situation: Some pullback P_2 of P_1 under f^{k_1} surrounds the critical point. If again the preimage Q_2 (being a preimage of Q_1 under f^{k_1}) surrounded by P_2 also surrounds the critical point we are done and have found two descendants P_2 and Q_2 to P and Q respectively. However, note that, whereas Q_2 is a child to Q, we have that P_2 is a child of P_1 and P_1 is a child of Q. Q_1 is not a child of Q since Q was assumed to be disjoint from the critical point.

Continuing in this way we find two descendants P_m and Q_m such that

$$f^{k+k_1+\ldots+k_{m-1}} : P_m \to P$$

as a 2^m degree unbranched covering and

$$f^{k+k_1+\ldots+k_{m-1}} : Q_m \to Q$$

as a 2 degree unbranched covering.

Hence, Q_m is a child to Q, whereas every P_{j+1} is a child to P_j, $j = 0, \ldots, m-1$. In this case we call the annulus between P_m and Q_m an *offspring* of α. Hence, every offspring has modulus at least 2^{-m} times the modulus of its *ancestor* α, where m is as defined above. Again, there cannot be any degenerate annulus A_j between P_m and Q_m. Otherwise, we could map this annulus forward: $f^{k+k_1+\ldots+k_{m-1}}(A_j)$ would be a degenerate annulus between P and Q.

Conversely, let P_m and Q_m be given degenerate annuli surrounding the complementary annulus α_1 and assume that there is no other degenerate annulus between P_m and Q_m. If Q_m has generation more than 1 then the parent Q would have generation at most 1. On the other hand, the parent P to P_1, which in turn is parent to P_2 and so on down to P_m, might have negative generation, meaning that P is actually a parent to A_0. In this case, A_0 would lie between P and Q. But in this case there has to be some preimage of A_0 laying between P_m and Q_m. This contradict the fact that there is no degenerate annulus between P_m and Q_m.

We conclude from the above discussion:

Lemma 7.12. *Every complementary annulus α between two degenerate annuli P and Q, where the generation of Q is larger than* 1*, has some unique ancestor β.*

Definition 7.1. Given a complementary annulus α surrounded by the outer degenerate annulus $A_{m,*}$ and the inner annulus $A_{n,*}$, we say that the *outer generation* to α is equal to m and the *inner generation* to α is m. We write $\alpha = \alpha^m_{n,*}$, where $*$ means an index, since there might be many α with the same m and n.

We have proved the following.

Lemma 7.13. *For every complementary annulus $\alpha = \alpha^m_{n,*}$ with $n > 1$ and with ancestor $\alpha^{m_1}_{n-1,*}$ we have*

$$\operatorname{mod}(\alpha^m_{n,*}) \geq 2^{m_1-m} \operatorname{mod}(\alpha^{m_1}_{n-1,*}).$$

Corollary 7.14. *For every complementray annulus $\alpha^m_{n,*}$, $n > 1$, there is some "grand" ancestor $\alpha^{m_{n-1}}_{1,*}$ such that*

$$\operatorname{mod}(\alpha^m_{n,*}) \geq 2^{m_{n-1}-m} \operatorname{mod}(\alpha^{m_{n-1}}_{1,*}).$$

Since the number of degenerate annuli of generation m is at least 2^m we have that the number of complementary annuli of outer generation m is at least 2^m. Moreover, trivially, we have $\operatorname{mod}(\alpha^m_{1,*}) \geq M_0$ for all m, for some $M_0 > 0$.

By Corollary 7.14 the sum of the moduli of all $\alpha^m_{n,*}$ for fixed m is at least

$$\sum_{n,*} \operatorname{mod}(\alpha^m_{n,*}) \geq 2^m 2^{-m} \operatorname{mod}(\alpha^{m_{n-1}}_{1,*}) \geq M_0.$$

Hence

$$\sum_{m,n,*} \operatorname{mod}(\alpha^m_{n,*}) = \infty,$$

and Lemma 7.11 follows.

7.4. **Combinatorics of the puzzle.** We make some definitions first. Let a_1, a_2 be two parameters in the same wake W. We say that R_{a_1} and R_{a_2} have the *same combinatorics of the puzzle up to depth* d if there exists an orientation preserving homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the following holds:

- ϕ homeomorphically maps distinct puzzle pieces P^i_k of depth $k \leq d$ of R_{a_1} to distinct puzzle-pieces Q^j_k of depth k of R_{a_2};
- for all $k \leq d$ we have $\phi : P_k(-1) \mapsto Q_k(-1)$;
- finally, ϕ *respects the dynamics*, that is,

$$P^i_k = R_{a_1}(P^j_k) \text{ if and only if } \phi(P^i_k) = R_{a_2}(\phi(P^j_k)).$$

Similarly, we will say that a quadratic polynomial f_c and R_a have the same combinatorics of the puzzle up to depth d, if there exists an orientation-preserving continuous surjection ϕ which maps puzzle-pieces of f_c to those of R_a up to depth d, sending critical pieces to critical ones, and respecting the dynamics.

Proposition 7.15. *Let f_c be a quadratic polynomial without non-repelling fixed points. For every d there exists a parameter a such that R_a and f_c have the same combinatorics of the puzzle down to depth d. Moreover, consider the puzzle-piece $P_d(c)$ of f_c, and let $\beta_1, \ldots, \beta_k$ be the angles of external rays which bound it. Then bubble rays with the same angles bound the puzzle piece $Q_d(-a)$ of R_a.*

Finally, there exists an open set Δ_d in the $a-$plane, with $\Delta_d \subset \Delta_{d-1}$, and $\Delta_0 = W$ such that R_b has the same combinatorics of the puzzle to depth d and $-b$ is contained in the particular puzzle piece of level d if and only if $b \in \Delta_d$.

Proof. The Proposition follows by a straightforward induction on the depth d. The base of induction, with $d = 0$ is given by Lemma 6.11. Assuming the statement is true at depth $d-1$, consider the pullback of the puzzle of level 1 inside the critical value piece $P_{d-1}(-a)$. By assumption, this picture has the same combinatorial structure as the similar one for f_c. By Lemma 6.7, as the parameter a moves through Δ_{d-1}, the critical value sweeps out $P_{d-1}(-a)$. We can hence select a parameter a to match the combinatorics of the puzzle of f_c down to level d. The parameter plane statement follows from similarly obvious consideration and is left to the reader.

□

Definition 7.6. We call a set Δ_d as above *a parameter puzzle piece.*

8. Existence of a Mating

Fix a Yoccoz' polynomial f_c which is not critically finite, non-renormalizable, and such that c does not belong to the 1/2-limb of the Mandelbrot set. By Proposition 7.15, there exists a parameter value a such that R_a has the same combinatorics of the puzzle as f_c for all $d \in \mathbb{N}$.

Lemma 8.1. *Consider any $z \in J(R_a)$ which is not a preimage of α_a or p_a. Then the nested sequence of puzzle pieces $P_d(z)$ shrinks to z:*

$$\bigcap P_d(z) = \{z\}.$$

Proof. Assume first that there exists some $N > 0$ such that the orbit of z is disjoint from $P_N(-1)$. In this case, the claim is implied by Lemma 7.10.

In the opposite case, for each $n \in \mathbb{N}$ consider the first instance i such that $R_a^i(z) \in P_{n+1}(-1)$. Then the complementary annulus $\alpha_{n+i}(z)$ is a conformal copy of $\alpha_n(-1)$. By construction, all these annuli around z are disjoint, and hence by Lemma 7.11,

$$\sum \operatorname{mod} \alpha_n(z) = \infty.$$

By Lemma 7.2, we have the claim.

□

Lemma 8.2. *Every bubble ray for R_a lands.*

Proof. This is obviously true for the preimages of the rays landing at the fixed point α. Let z be an accumulation point of any other ray $\mathcal{B} = \cup_0^\infty F_i$. There is an infinite sequence of nested puzzle pieces $P_d(z)$ containing z, and by the previous Lemma,

$$\bigcap P_d(z) = \{z\}.$$

Now by Lemma 2.7 the bubbles F_i do not cross the boundaries of $P_d(z)$, and hence

$$F_i \to z.$$

□

8.1. **Construction of semiconjugacies.** Consider the conjugacy

$$\phi : \overset{\circ}{K}_{\circ\bigcirc\circ} \cup_n f_{\circ\bigcirc\circ}^{-n}(\alpha) \mapsto \cup_n R_a^{-n}(A_\infty \cup \{p\})$$

defined in Proposition 4.5. By Lemma 8.2 and Lemma 5.6 it extends by continuity to a semi-conjugacy $K_{\circ\bigcirc\circ} \to \overline{\cup R_a^{-n}(A_\infty)} = \hat{\mathbb{C}}$:

$$\phi_1 \circ f_{\circ\bigcirc\circ}(z) = R_a \circ \phi_1(z). \tag{8.1}$$

Let $z \in J_c$ and not a preimage of α, and let $P_d(z)$ be the sequence of Yoccoz' puzzle-pieces of depth d containing z. Let $Q_d(z)$ be the corresponding pieces in the puzzle of R_a and define

$$\phi_2(z) = \cap Q_d(z).$$

By construction, ϕ_2 extends continuously to $\cup_n f_c^{-n}(\{\alpha\})$ and for the extended map

$$\phi_2 \circ f_c = R_a \circ \phi_2.$$

Let $\sim_r$ denote the ray equivalence relation generated by the quadratics $f_{\circ\bigcirc\circ}$ and f_c. We proceed to demonstrate:

Theorem 8.1. *We have*

$$\phi_i(z) = \phi_j(w)$$

if and only if they are in the same ray equivalence class,

$$z \sim_r w.$$

We begin with the following definition.

Definition 8.1. For $q > 1$, let

$$\theta_1 \mapsto \theta_2 \mapsto \cdots \mapsto \theta_q \mapsto \theta_1$$

be a period q orbit of the doubling map. The angles θ_i partition the circle into arcs A_i, $i = 1, \ldots, q$, which we enumerate in the counter-clockwise order starting from the arc containing 0. For $\theta \in \mathbb{T}$ which does not eventually fall into the orbit under doubling, we denote $\sigma_{\theta_1,\ldots,\theta_q}(\theta)$ the *itinerary* of θ with respect to the partition A_i, viewed as an infinite string in $\{1, \ldots, q\}^\infty$. In the case when θ is a preimage of one of the θ_i the itinerary $\sigma_{\theta_1,\ldots,\theta_q}(\theta)$ will be a finite string of digits between 1 and q – to avoid ambiguity, the last A_i will be chosen to the right of θ_i.

In a very similar way, let us define a symbol sequence $\sigma(z) \in \{1, \ldots, q\}^\infty$ with respect to the initial Yoccoz puzzle for f_c or the initial Yoccoz bubble-puzzle for R_a as follows. Enumerate the initial puzzle-pieces of f_c as P_0^k, $k = 1, \ldots, q$ in counter-clockwise order around α, starting with $P_0^1 \ni 0$. Set Q_0^k to be the puzzle piece of R_a, which corresponds to P_0^k. Put

$$\sigma(z) = \begin{cases} k & \text{if } f_c^j(z) \in P_0^k, \text{ for } z \in J(f_c) \setminus \cup_n f_c^{-n}(\alpha), \\ k & \text{if } R_a^j(z) \in Q_0^k, \text{ for } z \in J(R_a) \setminus \cup_n R_a^{-n}(\alpha_a \cup p_a). \end{cases}$$

Since ϕ_1 is a semi-conjugacy the following lemma is immediate.

Lemma 8.3. *Assume that $z \in K_{\circ\bigcirc\circ}$ is uni-accessible and let $\phi_1(z) = \zeta$. Let $-\beta$ be the angle of the external ray landing at z. If z is not a preimage of the α-fixed point, then*

$$\sigma(\zeta) = \sigma_{-\theta_1,\ldots,-\theta_q}(-\beta).$$

Recall now, that a point in the Julia set $J_{\circ\bigcirc\circ}$ is bi-accessible if and only if it is a preimage of $\alpha_{\circ\bigcirc\circ}$. The latter is the landing point of two external rays, $R_{1/3}$, and $R_{2/3}$, forming a period 2 cycle. Let d be the function $d : z \mapsto 2z \bmod \mathbb{Z}$.

Lemma 8.4. *Let R_θ be a ray landing at a bi-accessible point $x \in J_{\circ\bigcirc\circ}$. Then the landing point of $R_{-\theta}$ in J_c is uni-accessible.*

Proof. The angle $-\theta$ has a finite orbit under the doubling, and hence the orbit of the landing point y of the ray $R_{-\theta}$ is also finite. By assumption, f_c is not critically finite, and hence the orbit of y does not include 0. Denote n the first iterate for which $d^n(-\theta) \in \{1/3, 2/3\}$, and $z = f^n(y)$. Since f^n is a local homeomorphism on a neighborhood of y, the number m of accesses is the same for y and z. Assume that $m > 1$.

Note first that z cannot be a fixed point, as otherwise the ray portrait $\{\{1/3, 2/3\}\}$ is realized for f_c, and c is in the 1/2-limb. Hence z has period 2. By the properties of periodic external rays all rays landing at z have the same period, 2, and same for $f(z)$. Hence, there are $m \times 2 \geq 4$ angles in $\mathbb{T}$ whose period under the doubling is equal to 2. By inspection, 1/3 and 2/3 are the only angles with this property, and we have arrived at a contradiction. $\square$

By assumption, there exists $q > 2$ such that there is a cycle of rays $R_{\theta_1}, \ldots, R_{\theta_q}$ landing at the dividing fixed point α of f_c. By construction, a cycle of bubble rays $\mathcal{B}_{\theta_1}, \ldots, \mathcal{B}_{\theta_q}$ with the same angles lands at the fixed point α_a.

Lemma 8.5. *We have*

$$\phi_1(z) = \phi_1(w) \text{ if and only if } z \sim_r w.$$

Proof. By Lemma 2.7, only uni-accessible points can be identified. From Lemma 8.4 the lemma now follows if at least one of z and w is bi-accessible. Hence we can assume that both z and w are either landing points of infinite bubble rays $\mathcal{B}_1, \mathcal{B}_2 \subset K_{\circ\bigcirc\circ}$, or that one of z and w or both lies on a uni-accessible point on the boundary of a bubble. Denote $-\beta_1, -\beta_2$ the angles of the external rays landing at z and w respectively. (In the case when z and w are landing points of infinite bubble rays $\mathcal{B}_i$, note by definition, that the angles of these bubbles rays are β_1 and β_2 respectively.)

By Lemma 8.3, $\phi_1(z) = \phi_1(w)$ if and only if

$$\sigma_{-\theta_1,\ldots,-\theta_q}(-\beta_1) = \sigma_{-\theta_1,\ldots,-\theta_q}(-\beta_2). \tag{8.2}$$

Now, consider the external rays R_{β_i} of f_c. Since the combinatorics of the puzzles of f_c and R_a is the same for every depth, these two rays have a common landing point if and only if (8.2) holds. The statement of the lemma now follows from Lemma 8.4. $\square$

Lemma 8.6. *We have*

$$\phi_2(z) = \phi_2(w) \text{ if and only if } z \sim_r w.$$

Proof. Note that by Lemma 8.4, if $z \neq w$, then $z \sim_r w$ if and only if both of these points are uni-accessible, and denoting β_1, β_2 their external angles, we have $d^n(\beta_1) = 1/3$, $d^n(\beta_2) = 2/3$ for some n.

On the other hand, if $\zeta = \phi_2(z) = \phi_2(w)$, then $\zeta \in R_a^{-n}(p_a)$ for some n.

It is thus enough to show, that $\phi_2(z) = \phi_2(w) = p_a$ if and only if z, w are the landing points of the external rays $R_{1/3}$, $R_{2/3}$ respectively. By construction, at most two points in J_c are mapped to p_a by ϕ_2, so we only need to prove the second implication.

The landing points z, w of rays $R_{1/3}$, $R_{2/3}$ form a cycle of period 2, hence, the period of the cycle $\zeta_1 = \phi_2(z)$, $\zeta_2 = \phi_2(w)$ is at most 2. By Lemma 2.7, these points do not lie in the boundary of any of the bubbles. Assume that $\zeta_1 \neq p_a \neq \zeta_2$. Then there exists a bubble ray of angle θ landing at ζ_1. Since the combinatorics of the puzzle is the same for R_a and f_c,

$$\sigma_{\theta_1,\ldots,\theta_q}(\theta) = \sigma_{\theta_1,\ldots,\theta_q}(1/3).$$

This bubble ray then lands at a point in $J_{\circ\bigcirc\circ}$ with the external angle $2/3$, which is a contradiction.

□

We finish the proof of Theorem 8.1 with the following:

Lemma 8.7. *We have $\phi_1(z) = \phi_2(w)$ if and only if $z \sim_r w$.*

Proof. If $z \in K_{\circ\bigcirc\circ}$ is uni-accessible then let $-\beta$ be the angle of the external ray landing at z and put $\zeta = \phi_1(z)$. By Lemma 8.3,

$$\sigma_{-\theta_1,\ldots,-\theta_q}(-\beta) = \sigma(\zeta).$$

If $\zeta = \phi_2(w)$, then w lies in the same puzzle-pieces as the point ζ, by definition. An external ray R_γ (there can be more than one) which lands at w must by Lemma 8.3 satisfy

$$\sigma_{\theta_1,\ldots,\theta_q}(\gamma) = \sigma(\zeta).$$

Obviously, one solution is $\gamma = -\beta$, and therefore $z \sim_r w$. Conversely, if $z \sim_r w$, then $\phi_1(z) = \phi_2(w)$ by construction.

If $z \in K_{\circ\bigcirc\circ}$ is bi-accessible then the lemma follows from Lemma 8.4.

□

We conclude:

Main Theorem, the existence part. *Suppose c is a non-renormalizable parameter value outside the $1/2$-limb of $\mathcal{M}$. Then the quadratic polynomials f_c and $f_{\circ\bigcirc\circ}$ are conformally mateable.*

9. Uniqueness of mating

To transfer the results of shrinking puzzle pieces in the dynamical plane to the parameter plane, we use a variation of the approach of Yoccoz (see [**Hub**]). Our arguments follow the presentation of [**Ro1**].

Let us recall the following definition.

Definition 9.1. Let X be a connected complex mainfold. A *holomorphic motion* over a set $E \subset \mathbb{C}$ is a function

$$\varphi : X \times E \to \hat{\mathbb{C}},$$

where $\varphi(\lambda, z)$ is holomorphic in the variable $\lambda \in X$ and injective in $z \in E$.

We make use of a stronger version of the λ-lemma of Manẽ-Sud-Sullivan [**MSS**], due to Slodkowski [**Slo**].

The λ-Lemma. *A holomorphic motion over a set E has a unique extension to a holomorphic motion over $\overline{E}$. The extended motion gives a contiuous map $\varphi : X \times \overline{E} \to \hat{\mathbb{C}}$. For each $\lambda \in X$, the map $\varphi_\lambda : \overline{E} \to \hat{\mathbb{C}}$ extends to a quasiconformal map of $\hat{\mathbb{C}}$ to itself.*

Let us fix a parameter c satisfying the conditions of the Main Theorem. Let Δ_n be the nested sequence of parameter puzzle-pieces of Proposition 7.15 in the a-plane. Our aim is to show:

Theorem 9.1. *We have*

$$\operatorname{diam}(\Delta_n) \to 0.$$

Let us fix a parameter $a_0 \in \cap \Delta_n$. Let P be a parabubble intersecting some Δ_n. Denote B_a the bubble containing the critical value $-a$ for R_a with $a \in P$. Let $k \in \mathbb{N}$ be the smallest such that for any $a \in P$, $R^k(-a) \in A^a_\infty$. Let

$$\Phi_a : A^a_\infty \to \hat{\mathbb{C}} \setminus \mathbb{D}$$

denote the normalized Böttcher coordinate at infinity. By Lemma 5.6, it extends homeomorphically to the boundary. We then obtain a homeomorphism $P \mapsto B_{a_0}$ by the formula.

$$F : a \mapsto R_{a_0}^{-k} \circ \Phi_{a_0}^{-1} \circ \Phi_a \circ R_a^k(-a).$$

Pasting these homeomorphisms together, we obtain

Lemma 9.1. *There is a homeomorphism from the closure of the boundary of the parameter puzzle piece of depth n into the closure of the boundary of the puzzle of depth n for R_{a_0}.*

We now construct a holomorphic motion on the boundary of the puzzle at an initial level.

Lemma 9.2. *There is a holomorphic motion $h_n : \Delta_n \times I^{a_0}_{n+1} \to \hat{\mathbb{C}}$, where $I^{a_0}_{n+1}$ is the closure of the boundary of the puzzle of depth $n+1$. We have $h^a_n(I^{a_0}_{n+1}) = I^a_{n+1}$. Moreover, $R_a \circ h^a_n(z) = h^{a_0}_n \circ R_{a_0}(z)$, for any $z \in I^{a_0}_{n+1}$.*

Proof. Indeed, as a varies throughout Δ_n, the critical value does not hit the bubble rays corresponding to the puzzle of depth n according to Lemma 5.5. We get from Lemma 5.9 that A^a_∞ moves holomorphically on Δ_n. So do the preimages of A^a_∞ as long as we do not hit the critical value. It follows that every bubble B in the boundary of the puzzle of depth n moves holomorphically according to the formula

$$\eta_a(z) = R_a^{-k} \circ \Phi_a^{-1} \circ \Phi_{a_0} \circ R_{a_0}^k(z), \tag{9.1}$$

where k is smallest integer such that $R^k_a(z) \in A_\infty$, for $z \in B$.

Since the critical value does not intersect the puzzle of depth n, we can pull back this puzzle once so that the puzzle of depth $n+1$ moves holomorphically as well.

The λ-Lemma now extends the motion to its closure. It follows from (9.1) that $h^a_n(I^a_{n+1}) = I^{a_0}_{n+1}$ and that the diagram

$$\begin{array}{ccc} I^{a_0}_{n+1} & \xrightarrow{h^a_n} & I^a_{n+1} \\ R_{a_0}\big\downarrow & & \big\downarrow R_a \\ I^{a_0}_n & \xrightarrow{h^a_{n-1}} & I^a_n \end{array}$$

is commutative. $\square$

Definition 9.1. Let D^a_{n+1} be the puzzle piece bounded by $h^a_n(\partial P^{a_0}_{n+1})$, where $P^{a_0}_{n+1}$ is the puzzle piece P_{n+1} surrounding the critical value $-a_0$ at depth $n+1$.

We have the following:

Lemma 9.3. *The parameter $a \in \Delta_m \setminus \Delta_{m+1}$ if and only if the critical value $-a \in D^a_m \setminus D^a_{m+1}$.*

Proof. Take a non self-intersecting path a_t from a_0 to the boundary of Δ_m ,$t \in [0,1]$, crossing the boundary of Δ_{m+1} exactly once. Then the critical value $-a_t$ has to cross the boundary of $h^{a_t}_m(\partial P^{a_0}_{m+1})$, since we always have $D^a_m \supset D^a_{m+1}$. Assume this happns at $t = t_0$. Then for $t > t_0$ we get that $-a_t \notin D^{a_t}_{m+1}$, since we are outside Δ_{m+1}. Similarily, $-a_t \in D^{a_0}_{m+1}$ for $t < t_0$. □

Proof of Theorem 9.1. Let us first handle the harder case, when the critical tableau of f_c is recurrent.

Extend the holomorphic motion on Δ_{m_0} at depth m_0 by the λ-Lemma, so that we get a holomorphic motion on Δ_{m_0} with dilatation $K = K(\delta(a, \partial\Delta_{m_0}))$, which depends on the conformal distance $\delta(a, \partial\Delta_{m_0})$ from a to the boundary of Δ_{m_0}. Let us call this extended motion $\tilde{h}_{m_0}$.

Now, lift the motion $\tilde{h}_{m_0}$ via the unbranched covering maps $R_a^{m_j - m_0}$ for $a \in \Delta_{m_j}$. We get a holomorphic motion

$$\tilde{h}_{m_j} : \Delta_{m_j} \times A^{a_0}_{m_j} \longrightarrow \hat{\mathbb{C}},$$

where $A^{a_0}_{m_j} = P_{m_j}(-a_0) \setminus P_{m_j+1}(-a_0)$ is an annulus surrounding the critical value (the A_{m_j} are children to A_{m_0}). Since holomorphic composition does not change the dilatation, it follows that this lifted motion has the same dilatation K as $\tilde{h}_{m_0}$. Moreover, the annuli $A^{a_0}_{m_j}$ move holomorphically; set $A^a_{m_j} = \tilde{h}_{m_j}(A^{a_0}_{m_j})$. In other words, $A^a_{m_j} = D^a_{m_j} \setminus D^a_{m_j+1}$.

By Lemma 9.3, we have that $a \in \Delta_{m_j} \setminus \Delta_{m_j+1}$ if and only if $-a \in A^a_{m_j}$.

Define the *parameter annuli* $\mathcal{A}_n = \Delta_n \setminus \Delta_{n+1}$.

Fix the number $N = m_j$ from now on and let $\Delta_N = \Delta$. Define a map defined on Δ, by

$$H = H_N : a \mapsto \tilde{h}_a^{-1}(-a).$$

We see that $H_N : \mathcal{A}_N \to A^{a_0}_N$. On the boundary of Δ it is injective, which follows directly from Lemma 5.9.

The next issue is to show that the map H_N is quasiconformal with a definite bound on the dilatation independent of N. Here the proof is again the same as in [**Ro1**]; let us differentiate the relation $\tilde{h}^a_N(H_N(a)) = -a$. Then we get

$$\overline{\partial} h^a_N(H_N(a))\overline{\partial H_N(a)} + \partial h^a_N \overline{\partial} H_N(a) = 0.$$

This implies that the Beltrami coefficient $\mu(a) = \overline{\partial} H_N / \partial H_N$ satisfies

$$|\mu(a)| = \frac{|\overline{\partial} h^a_N(H_N(a))|}{|\partial h^a_N(H_N(a))|} = \frac{K_N - 1}{K_N + 1} < 1,$$

where K_N is the dilatation of h^a_N. However, if we consider the conformal representation $\chi : \Delta_N \mapsto \mathbb{D}$, The λ-Lemma implies that

$$K_N = \frac{1 + |\chi(a)|}{1 - |\chi(a)|}.$$

Since the sets Δ_{m_j} is compactly contained in Δ_{m_0} for $j \geq 2$, we get that $K_{m_j} \leq K$, for all $j \geq 2$.

We claim that the map H_N is injective. First of all, it is injective on the boundary of $\mathcal{A}_N$. Moreover, if we solve the Beltrami equation for μ, then we get a quasiconformal map $\phi : \mathcal{A}_N \to \phi(\mathcal{A}_N)$, so that $\overline{\partial}\phi = \mu\partial\phi$. It follows that $H_N \circ \phi^{-1}$ is conformal. By the Riemann- Hurwitz formula, there can not be any branch points in $\mathcal{A}_N$. Since H_N is injective on the boundary of $\mathcal{A}_N$, it follows that $H_N \circ \phi^{-1}$ maps $\phi(\mathcal{A}_N)$ conformally onto $A^{a_0}_N$. It follows that H_N must be a homeomorphism.

Since the annulus $A_{m_0}(-1)$ may be degenerate, we again consider the complementary annuli $\alpha_m(-1)$.

It follows that

$$\frac{1}{K} \operatorname{mod} \alpha^{a_0}_{m_j} \leq \operatorname{mod} \tilde{\alpha}_{m_j} \leq \frac{1}{K} \operatorname{mod} \alpha^{a_0}_{m_j},$$

where $\tilde{\alpha}_m$ denotes a complementary annulus in the parameter plane. Since

$$\sum \operatorname{mod} \alpha_N = \infty \text{ we have } \sum \operatorname{mod} \tilde{\alpha}_N = \infty,$$

and we conclude from Lemma 7.2 that the parameter pieces Δ_N shrink to a single point, which has to be a_0.

In the non-recurrent case, consider the puzzle of depth N so that the critical puzzle piece $P_N(-1)$ is disjoint from the postcrtical set. As the critical value $-a$ varies through Δ_N the puzzle at depth $N+1$ moves holomorphically as in Lemma 9.2. Hence every annulus $A_N(z)$ moves holomorphically. Extend this holomorphic motion by Slodkowski's Theorem and denote the extended motion by $\tilde{h}$ similar to the above argument. Since every annulus $A_n(-a_0)$, for $n > N$, is a univalent pullback of some $A_N(z)$ (since $R - a_0$ is non-recurrent) we can lift the holomorphic motion $\tilde{h}$ to the parameter piece Δ_n over $P_n(-a_0)$. Define a map $H_n : \mathcal{A}_n \mapsto A_n(-a_0)$ in exactly the same way as above. The proof of the fact that the parameter annuli shrink to a single point is now similar to the recurrent case and we leave the details to the reader.

□

We conclude:

Main Theorem, the uniqueness part. *The mating in Main Theorem is unique.*

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